A TREATISE

ON

PHYSICAL OPTICS

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BY

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PREFACE.

THE Science of Optics embraces so large a class of phenomena, that any treatise which attempted to give a comprehensive account of all the various practical applications of Optics, in addition to the experimental and theoretical portions of the subject, would necessarily be of an exceedingly voluminous character. I have accordingly limited the present work to one special branch of Optics, and have endeavoured to place before the reader as concise a treatise upon the Mathematical Theory of Light, and such experimental phenomena as are immediately connected therewith, as the nature of the case will admit.

Those who are acquainted with the Mathematical Theories of Hydrodynamics, Sound and Elasticity on the one hand, and of Electricity and Light on the other, cannot fail to have been struck with the difference, which exists between the two classes of subjects. In the former class, certain equations are obtained, which approximately, though not quite accurately, specify in a mathematical form, the physical state of fluids and solids as they exist in Nature; and the subsequent investigation of these branches of Science, is thereby in great measure reduced to a question of mathematics. As soon as the fundamental equations are established, the subject is brought within the dominion of mathematical analysis, and mathematicians are enabled to exercise their ingenuity and analytical skill, in elaborating and developing the results which flow from them.

But in the Theory of Light, we are confronted with a totally different state of things. Although the existence of the luminiferous ether may, at the present day, be regarded as a scientific axiom, which is as firmly established as any other scientific law, yet the properties of the ether are almost entirely unknown to us.

We are therefore unable to start with certain definite equations, of whose approximate correctness we can feel assured; but are compelled to formulate certain hypotheses concerning the physical constitution of the ether, which are capable of being expressed in a mathematical form, and then to trace the consequences to which they lead us. The only means at our disposal for testing the correctness of any hypothesis, which forms the foundation of any dynamical theory of light, is to compare the results furnished by theory, with known experimental facts; accordingly a knowledge of the facts, which it is the object of theory to explain, is of the utmost importance.

Under these circumstances, it has been necessary to describe at length, a variety of phenomena of an experimental character; but as the object of this work is to investigate the dynamical theory of light in relation to experimental phenomena, I have abstained from entering into many details respecting the methods of performing optical experiments, or the description of the necessary instrumental appliances. Those who desire a fuller acquaintance with the experimental portions of the subject, are recommended to consult Verdet's *Leçons d'Optique Physique*, Mascart's *Traité d'Optique*, and Preston's *Theory of Light*, where ample information concerning these matters will be found.

The first ten chapters are devoted to the consideration of Interference, Colours of Thick and Thin Plates, Diffraction, Double Refraction, Rotatory Polarization, and Reflection and Refraction of Polarized Light; and in these chapters, dynamical theories are as far as possible dispensed with. The remaining ten chapters are of a more speculative character, and contain an account of some of the dynamical theories, which have been proposed to explain optical phenomena. The investigations of numerous physicists upon the theories of Reflection and Refraction at the Surfaces of Isotropic and Crystalline Media, upon Double Refraction, Absorption, Anomalous Dispersion and Metallic Reflection are considered; and a description of a variety of experimental results, which require a dynamical theory to account for them, is given. The last two chapters are devoted to Maxwell's *Electromagnetic Theory*, together with the additions made to it since the death of its author. The last chapter of all, contains a description of the experimental results of Faraday, Kerr and Kundt on the action of electromagnetism on light; together with a development of Maxwell's theory, which is believed to be capable of accounting

for the action of a magnetic field, when light is propagated through a transparent medium.

It must be admitted, that some of the theories discussed in the later Chapters are of a somewhat speculative character, and will require reconsideration as our knowledge of the properties of matter increases; but at the same time, it is a great assistance to the imagination to be able to construct a mechanical model of a medium, which represents, even imperfectly, the action of ponderable matter upon ethereal waves. A full and complete investigation, by the aid of rigorous mathematical analysis, of the peculiar action of a medium, which is assumed to possess certain definite properties, enables us to understand the reason why certain effects are produced, and cannot fail to impress upon the mind the conviction, that the dynamical theory of light is a reality, rather than a scientific speculation. I have a profound distrust of vague and obscure arguments, based upon general reasoning instead of upon rigorous mathematical analysis. Investigations founded upon such considerations are always difficult to follow, are frequently misleading, and are sometimes erroneous. When the conditions of a dynamical problem are completely specified, all the circumstances connected with the motion can be expressed in mathematical language, and equations obtained, which are sufficient for the solution of every conceivable problem; and if the number of equations obtained is insufficient, the specification is incomplete, or some necessary condition has been overlooked. I consider it most important to the interests of mathematical physics, that the solutions of the various problems which present themselves, should be properly worked out by rigorous analysis, whenever it is possible to do so, and that definite mathematical results should be obtained and interpreted.

At the end of some of the earlier Chapters, examples and problems have been inserted, which have been derived from the examination papers set in the University of Cambridge. During recent years, there has been a disposition in certain quarters to question the educational value of examples; and a little sarcasm has occasionally been indulged in, with reference to so-called mathematical conundrums. One of the difficulties, which the Examiners for the Mathematical Tripos have to contend against, is the tendency on the part of Candidates to devote their time to learning certain pieces of book-work, which are likely to be set, instead of endeavouring to acquire an accurate and thorough

knowledge of the fundamental facts and principles of the subjects, which they take up; and well-selected examples and problems illustrating the book-work are of great assistance to an examiner, in enabling him to discriminate between candidates, who have acquired a perfunctory knowledge of a subject, and those who have endeavoured to master it. But it would be a fallacy to imagine, that the utility of examples and problems is exclusively confined to the particular examination in which they are set; or that the practical value of a problem is to be estimated solely by the scientific value of the result, which it embodies. The existence of a large collection of examples and problems, is of great assistance to future generations of students, in enabling them to grasp the fundamental principles of a subject, and to acquire facility in the application of mathematical analysis to physics; and the severe course of training, which the University of Cambridge exacts from students of the higher branches of Mathematics, is of inestimable benefit to Science, in producing a body of men, who are thoroughly conversant with dynamical principles, and are able to employ with ease the more recondite processes of mathematical analysis.

That the scientific discoveries of the present century have been of incalculable benefit to mankind, will be admitted by all; but it is also a distinct advantage to Science, when any discovery in abstract Science turns out to be of practical utility. The optical properties of chemical compounds have already been applied as a test of their purity; and it is probable, that further investigations of this character will be found to place a powerful weapon in the hands of Chemists.

I have to acknowledge the great assistance, which I have received from Verdet's *Leçons d'Optique Physique*, as well as from the original papers of the eminent mathematicians and experimentalists, which are referred to in the body of this treatise. I am also much indebted to Mr J. Larmor for having read the proof sheets, and for having made numerous valuable suggestions during the progress of the work.

April, 1892.

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,,	139	,,	5, *read* $\sin\alpha \sin(\alpha-\beta)$ *for* $\sin\alpha \sin(\alpha-\beta$.
,,	144	,,	12, *read* $\sin^4 i$ *for* $\sin^2 i$.
,,	242	,,	17, *read* κ *for* k.
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CHAPTER I.

INTRODUCTION.

1. THE Science of Optics may be divided into the following four distinct branches:—(i) Geometrical Optics, whose object is to investigate the laws relating to the reflection and refraction of light, and the theory of optical instruments; (ii) Experimental Optics, whose object is to discover the optical properties of transparent and other substances, which are capable of affecting light; (iii) Physical Optics, whose aim is to explain optical phenomena by means of a dynamical theory; (iv) Physiological Optics, which deals with the sensation produced by light upon the retina of the eye. The present treatise will be principally confined to the third branch of the subject; but inasmuch as the fundamental object of every theory of light is to explain experimental facts by means of dynamical principles, it will be necessary to describe and discuss a variety of experimental phenomena, in order that we may be in a position to compare the results furnished by theory, with those established by experiment.

2. Two theories of light have been proposed, which are of a totally different character; viz. the Corpuscular Theory and the Undulatory Theory. The corpuscular theory was proposed by Newton, and it assumes that a luminous body emits material particles in all directions, which, by their impact upon the retina, produce the sensation of light. By the aid of this theory, Newton succeeded in explaining the linear propagation, and also the reflection of light; but amongst other imperfections, the theory leads to the conclusion, that the velocity of light in highly refracting substances is greater than in substances of less refractive power, whereas it can be proved by experiment that the exact

converse is the case. The theory is also incapable of accounting for the polarization of light, and is now universally condemned as untenable.

The undulatory theory was first proposed by Huygens in 1678, and in the hands of Young, Fresnel and others, has been found capable of furnishing so satisfactory an explanation of experimental results, that it is now universally accepted as the true theory of light. The undulatory theory assumes, that all space is filled with a medium called the luminiferous ether, which is capable of being thrown into a state of vibration or wave motion, and of transmitting such vibrations with a definite velocity. Whenever any substance, which is capable of exciting periodic motion in the ether, exists in any portion of space, it is supposed that waves consisting of periodic vibrations are continually propagated in all directions. When the waves reach the retina of the eye, the ether in contact with it is set in motion, and a certain effect is produced upon the retina, which is transmitted to the brain by means of the optic nerve, and gives rise to the sensation of light.

3. It must not however be supposed, that all vibrations which the ether is capable of executing, necessarily affect the eye. In the Theory of Sound, which presents many points of analogy with the Theory of Light, it is well known that it is extremely easy to produce a disturbance in the air, which possesses all the physical properties of a wave of sound, but which nevertheless is incapable of producing any impression upon the ear. In fact the ear is only capable of taking cognizance of waves of sound, whose periods lie within certain definite limits. Similarly the eye is only capable of being affected by ethereal waves, whose periods lie between certain limits, which are far narrower than the corresponding limits in the case of Sound, and which in acoustical language may be described as lying within an octave. At the same time it is certain that ethereal waves exist, whose periods lie outside the limits of the extreme red and violet rays of the spectrum, which possess all the physical properties of waves of light, but whose existence can only be discovered by the thermal or chemical effects which they produce.

The luminiferous ether is supposed to exist not only in air, liquids, and ultraterrestrial space, but also in solid bodies. Molecular theories furnish strong grounds for the conclusion, that the

molecules of even the densest and hardest bodies do not form an absolutely compact mass, but that interstices exist between them, which are filled with ether; and it is owing to this cause, that hard transparent substances like glass and diamond are capable of transmitting light. If therefore we assume that a hard and dense substance, such as glass, contains interstices which are filled with ether, it would be somewhat inconsistent to suppose, that the opacity of a soft substance, like tallow, is due to the fact, that it does not contain ether. We must therefore look for an explanation of the opacity of tallow or wax, as being due to some peculiar action of the molecular structure of such materials, which prevents the ether contained in them from transmitting vibrations.

If we were fully acquainted with the physical properties of the ether and its relation to ponderable matter, the explanation of optical phenomena would become a mere question of mathematical analysis. We should simply have to translate the physical properties of the ether into mathematical language, and should thereby obtain the equations of motion and the boundary conditions; and the solution of these equations would furnish a complete theoretical explanation of every experimental result. But although there are abundant grounds for justifying our belief in the existence of the ether, we are almost completely in the dark respecting its properties. Accordingly, the only means at our disposal for obtaining any information upon this question, is to adopt the inductive method of making some hypothesis which is dynamically sound and physically possible, and which is capable of being expressed in a mathematical form, and then to compare the results to which theory leads us with known experimental facts. If the results of our theory are wholly inconsistent with experiment, the theory must be abandoned; but if our theoretical results are wholly or partially in accordance with experiment, we shall be justified in concluding that our theory, if not absolutely true, may at any rate contain a germ of truth.

4. The first rigorous dynamical theory of light was proposed by Green in 1837. He supposed that the ether is a medium which is capable of resisting compression and distortion; and he showed that when the medium is isotropic, the equations of motion are the same as those of an isotropic elastic solid, and contain two independent constants, one of which measures the resistance to compression, and the other the resistance to distortion. Green

was also led to assume from physical considerations, that the first constant is very large in comparison with the second. This has sometimes been interpreted to mean that the ether is almost incompressible, but an assumption of this kind is not an essential part of Green's theory. The theory would be satisfied, if the ether were more compressible than the most highly compressible gas; all that is necessary is, that the *ratio* of the resistance to compression to the resistance to distortion should be very large. Green's theory explains fairly well the propagation of light in isotropic media, but it fails to furnish a satisfactory theory of double refraction, and takes no account of rotatory polarization and dispersion.

5. The second class of theories, are theories based upon the mutual reaction of ether and matter. When a wave of light impinges upon a transparent or opaque substance, it is supposed that the vibrations of the ether set the molecules of the matter composing the substance into a state of vibration, and that these vibrations modify the motion of the ether. If the ether be regarded as a substance possessing a density, which is *finite* though excessively small compared with the densities of substances ordinarily met with, the action of the matter will produce certain forces which affect the ether. It has been proved experimentally, that if the light which is emitted from certain substances when incandescent (such as burning sodium), be transmitted through the vapour of those substances, light will be absorbed. The explanation of this phenomenon, which was first suggested by Stokes[1], depends upon a theorem due to Sir J. Herschel, *that when a dynamical system is acted upon by a periodic force, whose period is equal, or nearly so, to one of the periods of the free vibrations of the system, the corresponding forced vibration will be large.* Now sodium vapour when incandescent, emits light of a certain definite period, which is consequently one of the free periods of the vibrations of sodium vapour. Accordingly, when light from a sodium flame passes through sodium vapour, the molecules of the vapour are thrown into a violent state of vibration; and as the energy required for the maintenance of these vibrations must be supplied by the waves of light, it follows that a comparatively small portion of the energy which enters the vapour emerges from it, and therefore light is absorbed. The

[1] *Phil. Mag.* March, 1860, p. 196.

foregoing illustration will give some idea of the way in which the vibrations of ether are affected by the presence of matter.

6. The third theory, is the electromagnetic theory due to the late Professor J. Clerk-Maxwell, which supposes that light is the effect of an electromagnetic disturbance. This theory not only explains the propagation of light in isotropic media, but furnishes the most satisfactory theory of double refraction which has yet been proposed.

The early death of Maxwell at the age of forty-nine prevented him from elaborating and completing this theory, which as it left his hands, took no account of dispersion, nor of rotatory polarization produced by quartz or turpentine; but several important additions to the theory have been made during recent years, which will be considered in the concluding chapters of this work.

7. To commence this treatise with a discussion of the various theories of light, which have been briefly touched upon in the foregoing paragraphs, would throw a considerable strain upon the mathematical resources of the reader; but inasmuch as there are a variety of optical phenomena, which are capable of being explained upon the hypothesis of a medium which is capable of propagating waves, without entering into any speculations concerning the physical properties of the medium, I think that the best course to adopt will be to dispense as far as possible with dynamical theories for the present, and to endeavour to explain the phenomena which present themselves by means of the geometrical properties of wave motion. We shall thus be able to explain Interference, Colours of Thick and Thin Plates, Diffraction, and Polarization. Double Refraction cannot be satisfactorily discussed without a theory of some kind; but having given an account of Fresnel's theory, which although dynamically unsound, is of great historical interest, we shall be able to investigate the geometrical properties of Fresnel's wave surface, and the production of coloured rings by thin crystalline plates; and we shall then be in a better position to understand the more theoretical portions of the subject.

8. When a source of light exists in an isotropic medium, spherical waves concentric with the source are propagated throughout the medium; and if the effect, which these waves produce at some portion of space, whose greatest linear dimension is small in

comparison with its distance from the source, be observed, the waves may be regarded as approximately plane. We are thus led in the first instance to study plane waves.

Let us therefore suppose, that a train of plane waves is propagated in some given direction, which we shall choose as the axis of x; then if v be the displacement of the medium, we may write

$$v = A \cos \frac{2\pi}{\lambda} (x - Vt - e) \text{ .... (1).}$$

A discussion of the kinematical properties of wave motion is given in my *Elementary Treatise on Hydrodynamics and Sound*, § 76, to which the reader is referred. It will be observed, that the right-hand side of (1) contains four constants A, V, λ, e; we may also if we please introduce the period τ in the place of V or λ, because $V\tau = \lambda$; accordingly (1) may be written in the forms

$$v = A \cos 2\pi \left(\frac{x-e}{\lambda} - \frac{t}{\tau}\right) = A \cos \frac{2\pi}{\tau} \left(\frac{x-e}{V} - t\right) \text{...... (2),}$$

which are often useful.

9. We must now enquire, how these four constants are related to the physical properties of a wave.

The quantity e is called the phase of the wave, and fixes its position with respect to the point assumed as the origin. This constant is therefore a purely geometrical one.

The quantity V is called the velocity of propagation of the wave, and measures the velocity with which light is travelling in the medium. In isotropic media, V is independent of the direction of the wave, but in æolotropic media, V is a function of the direction.

That light travels with a finite velocity, was first established in 1676 by the Danish astronomer Olaus Römer, who observed that when the planet Jupiter was nearest to the Earth, the eclipses of Jupiter's moons happened earlier than they ought to have done according to the astronomical tables; whilst when Jupiter was farthest from the Earth, they happened later. He therefore concluded that the difference between the observed and calculated times was due to the fact, that light occupies a finite time in travelling from one point to another, and he calculated that the velocity of light in vacuo was 3023×10^5 metres per second.

More accurate measurements show that the velocity of light in vacuo is about 299,860,000 metres per second.

The velocity of light in vacuo, and in media which do not produce dispersion, is practically the same for all colours, and appears from theoretical considerations to depend upon the properties of the medium in which the light is being propagated. Both theory and experiment agree in showing that the colour of light depends upon the period of vibration; or, since the velocity is approximately independent of the colour, the colour may also be considered to depend on the wave-length. The periods of the red waves are the longest, whilst those of the violet are the shortest; accordingly the wave-lengths of the red waves are longer than those of the violet waves.

The imagination may be assisted by considering the analogous problem of waves of sound; for it is known that the velocity of different notes is very approximately the same, and that the pitch of a note is determined either by its period or its wave-length, notes of low pitch corresponding to waves of long period or large wave-length. There is however an important distinction between sound and light; for if the period of one note is double that of another, the notes stand to one another in the relation of octaves. Nothing corresponds to this in the case of light, and one reason of this is, that the sensitiveness of the ear extends over several octaves, whereas the sensitiveness of the eye is limited to less than an octave.

A table of the wave-lengths of the principal lines of the spectrum will hereafter be given; but it will be well to mention, that the wave-lengths, in tenth-metres, of the extreme red and violet rays are about 7604 and 3933 respectively. A tenth-metre is 10^{-10} of a metre.

10. The intensity of light has usually been considered to be proportional to the square of the amplitude. This may be seen as follows. Let O be a source of light, and let dS, dS' be elements of the surfaces of two spheres, whose common centre is O, which are cut off by a cone whose vertex is O. Let r, r' be the radii of the spheres, I, I' the intensities at dS, dS'. Since the total quantity of light which falls on the two elements is equal,

$$IdS = I'dS',$$

whence

$$Ir^2 = I'r'^2.$$

Now if A, A' be the amplitudes of the spherical waves at dS, dS', every dynamical theory of light shows, that A and A' are inversely proportional to r and r'; whence

$$I/A^2 = I'/A'^2,$$

from which we infer that the intensity is proportional to the square of the amplitude.

The undulatory theory supposes that a state of vibration is propagated through the ether; hence every element of volume possesses energy, which travels through space with the velocity of the wave. Modern writers have thus been led to the conclusion that an intimate connection exists between energy and intensity, but as regards the mathematical form of the connection between the two, opinions are not altogether uniform. Lord Rayleigh measures the intensity of light, *by the rate at which energy is propagated across a given area parallel to the waves*[1]; on the other hand, writers on the electromagnetic theory measure the intensity, *by the average energy per unit of volume.* Now so long as we are considering the propagation of light in a single isotropic medium, it is not of much consequence which definition we adopt, since the ratio of the intensities of two lights is proportional to the ratio of the squares of their amplitudes; but when we are considering the *refraction* of light, the ratio of the intensities of the incident and the refracted light will not be proportional to the squares of their amplitudes, but will contain a factor, whose value will depend upon which definition we adopt. It will be advantageous to adopt a definition, which will make the ratio of the intensities of the incident to the refracted light the same in the various dynamical theories which we shall hereafter consider, and I shall therefore define *the intensity of light to be measured by the average energy per unit of volume.*

It is well known, that when the medium is a gas or an elastic solid, the total energy due to wave motion is half kinetic and half potential; it can also be shown in the case of electromagnetic waves, that the total energy is half electrostatic and half electrokinetic. If therefore we assume that a similar proposition is true in the case of the ether, with regard to whose physical properties we have not at present made any

[1] "Wave Theory," *Encyclopædia Britannica.*

hypothesis, it follows from (2) that the total energy E per unit of volume is

$$E = \frac{4\pi^2 A^2 \rho}{\tau^2} \cos^2 \frac{2\pi}{V\tau}(x - e - Vt)$$

$$= \frac{2\pi^2 A^2 \rho}{\tau^2} \left\{1 + \cos \frac{4\pi}{V\tau}(x - e - Vt)\right\} \quad \ldots\ldots\ldots (3).$$

We therefore see that E consists of a non-periodic and a periodic term, the latter of which fluctuates in value. The former term represents what we have called the average energy per unit of volume, and accordingly the intensity is measured by the quantity $2\pi^2 A^2 \rho/\tau^2$, and is directly proportional to the product of the density and the square of the amplitude, and inversely proportional to the square of the period.

11. Up to the present time we have been considering the propagation of waves in an elastic medium, and have not made any supposition as to the *direction* of vibration. We now come to a point of capital importance, which constitutes one of the fundamental distinctions between waves of sound and waves of light. It is well known that in a plane wave of sound, the displacement is *perpendicular* to the wave front; but in the case of plane waves of light, the displacement lies *in* the front of the wave. This is established experimentally in the following manner. There are certain crystals, of which Iceland spar is a good example, which possess the power of dividing a given incident ray into two refracted rays, and from this property such substances are called doubly refracting crystals. Iceland spar is called a uniaxal crystal, owing to the fact that there is a certain direction, called the optic axis of the crystal, with respect to which the properties of the crystal are symmetrical. The symmetry of uniaxal crystals is therefore of the same kind as that of a solid of revolution. There is a second class of doubly refracting crystals, called biaxal crystals, which possess two optic axes, with respect to neither of which the properties of the crystal are symmetrical. Now if either of the two refracted rays, which are produced by a single ray incident upon a plate of Iceland spar, be transmitted through a second plate cut parallel to the axis, it will be found that although this ray is usually divided into two refracted rays, there are four positions, at right angles to one another, in which one or other of the two refracted rays is absent. If the second plate be placed in one of these four

positions, and then be turned round an axis perpendicular to its faces, the absent ray immediately appears, and its intensity increases, whilst that of the original ray diminishes; when the plate has been turned through a right angle, the intensity of the ray which was absent is a maximum, whilst the original ray has altogether disappeared; and when the second plate has been turned through two right angles, the original state of things is restored. Now if the vibrations were perpendicular to the wave front, and therefore in an isotropic medium parallel to the ray, the properties of a ray would be the same on every side of it; if however the vibrations were parallel to the wave front, and therefore perpendicular to the ray, we should anticipate that the properties of a ray would be different on different sides of it. We are thus led to the conclusion that the vibrations of light are parallel to the front of the wave, and this conclusion is amply justified by theory and experiment.

12. Let us now suppose that the axis of x is the direction of propagation, and that the axis of y is parallel to the direction of vibration; then if v be the displacement, it follows from (1) that

$$v = A \cos \frac{2\pi}{\lambda} (x - Vt - e) \qquad (4).$$

Equation (4) represents what is called a wave of *plane polarized light*, and the plane of xz, which is the plane to which the vibrations are perpendicular, is called the *plane of polarization.*

It was for many years a disputed point, whether the vibrations of polarized light were in or perpendicular to the plane of polarization, but modern investigations have shown that the latter supposition is the true one. We shall discuss this point in detail in subsequent chapters.

We shall now show how two trains of waves, whose wavelengths are equal, and whose planes of polarization are the same, may be compounded.

Let the first wave be given by (4), and let the second wave be

$$v' = A' \cos \frac{2\pi}{\lambda} (x - Vt - e');$$

then if $\qquad \phi = (2\pi/\lambda)(x - Vt - e), \ \delta = (2\pi/\lambda)(e - e');$

we obtain
$$\begin{aligned} v+v' &= A\cos\phi + A'\cos(\phi+\delta) \\ &= (A+A'\cos\delta)\cos\phi - A'\sin\phi\sin\delta \\ &= (A^2+A'^2+2AA'\cos\delta)^{\frac{1}{2}}\cos(\phi+\alpha)\ \ldots\ (5), \end{aligned}$$
where
$$\tan\alpha = \frac{A'\sin\delta}{A+A'\cos\delta}.$$

We therefore see that the two waves compound into a single wave, whose amplitude and phase are different from those of either of the original waves.

The amplitude is proportional to $(A^2+A'^2+2AA'\cos\delta)^{\frac{1}{2}}$, and may therefore vary from $A-A'$ to $A+A'$; the first of which corresponds to $\delta=(2n+1)\pi$, or $e=e'+(n+\frac{1}{2})\lambda$, and the second to $\delta=2n\pi$ or $e=e'+n\lambda$. We therefore see, that when the phases differ by an odd multiple of half a wave-length, the intensity of the light due to the superposition of the two waves is diminished; whilst when the difference of phase is an even multiple of half a wave-length, the intensity of the resultant light is increased.

If $A=A'$, we see that the intensity of the resultant light vanishes in the former case. Thus the superposition of two lights can produce darkness. This remark will be found to be of great importance in subsequent chapters.

13. We shall now consider the effect of compounding two waves which are polarized in perpendicular planes.

Let the two waves be
$$\left.\begin{aligned} v &= A\cos\frac{2\pi}{\lambda}(x-Vt-e) \\ w &= A'\cos\frac{2\pi}{\lambda}(x-Vt-e') \end{aligned}\right\}\ \ldots\ldots\ldots\ldots\ldots\ldots\ (6),$$
in which the first represents a wave polarized in the plane xz, and the second a wave polarized in the plane xy. In the notation of the last section, these equations may be written
$$\begin{aligned} v &= A\cos\phi, \\ w &= A'(\cos\phi\cos\delta - \sin\phi\sin\delta). \end{aligned}$$

Eliminating ϕ, we obtain
$$\frac{v^2}{A^2}+\frac{w^2}{A'^2}-\frac{2vw\cos\delta}{AA'} = \sin^2\delta\ \ldots\ldots\ldots\ldots\ldots\ (7).$$

From equation (7), we see that the elements of ether describe ellipses, whose planes are at right angles to the direction of

propagation. The resultant light in this case is said to be *elliptically polarized.*

It therefore follows, that elliptically polarized light may be produced by the composition of two waves of plane polarized light, whose amplitudes and phases are different, and whose planes of polarization are at right angles; and conversely, a wave of elliptically polarized light can always be resolved into two waves of plane polarized light, whose planes of polarization are at right angles.

If $A = \pm A'$, and $\delta = (n + \frac{1}{2})\pi$, or $e = e' + (\frac{1}{2}n + \frac{1}{4})\lambda$, equation (7) becomes

$$v^2 + w^2 = A^2.$$

Here the phases differ by an odd multiple of a quarter of a wave-length; in this case the elements of ether describe circles, and the light is said to be *circularly polarized.*

A circularly polarized wave may therefore be regarded as compounded of the two plane polarized waves

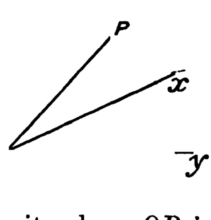
$$\left.\begin{aligned} v &= \pm A \cos \frac{2\pi}{\lambda}(x - Vt - e) \\ w &= A \sin \frac{2\pi}{\lambda}(x - Vt - e) \end{aligned}\right\} \quad (8).$$

Let us now suppose, that the observer is looking along the axis of x in the direction in which the light is being propagated, and let us take the upper sign in (8). If P be any element of the ether,

$$POy = (2\pi/\lambda)(x - Vt - e) = \phi;$$

accordingly,

$$\frac{dv}{dt} = \frac{2\pi A}{\tau} \sin \phi, \quad \frac{dw}{dt} = -\frac{2\pi A}{\tau} \cos \phi;$$



whence the resultant velocity along OP is zero; and if U be the resultant velocity perpendicular to OP, measured in the direction in which ϕ decreases,

$$U = \dot{v} \sin \phi - \dot{w} \cos \phi = 2\pi A/\tau.$$

Since U is positive, the direction of vibration is from the left-hand to the right-hand of the observer; and a wave of this kind is called a *right-handed circularly polarized wave*[1]. We see that it may be compounded of the two plane polarized waves

$$v = A \cos \phi, \quad w = A \sin \phi.$$

If the lower sign be taken, the angle $POy = \pi - \phi$; whence if U' be the velocity perpendicular to OP', measured in the same direction as before,

$$U' = \dot{v} \sin \phi + \dot{w} \cos \phi = -2\pi A/\tau.$$

Since U' is negative, the direction of vibration is from right to left; and a wave of this kind is called a *left-handed circularly polarized wave.* It may be compounded of the two plane polarized waves

$$v = -A \cos \phi, \ w = A \sin \phi.$$

Conversely, any plane polarized wave may be regarded as being compounded of a right-handed and a left-handed circularly polarized wave, whose amplitudes, phases and velocities of propagation are equal.

If $\delta = n\pi$, or $e = e' + \frac{1}{2}n\lambda$, (7) reduces to $v/A = w/A'$, which represents a plane polarized wave. From this result, we see that two plane polarized waves cannot compound into another plane polarized wave, unless their phases differ by a multiple of half a wave-length.

The methods by which plane, circularly, and elliptically polarized light can be produced, will be described in a subsequent chapter.

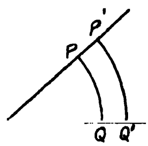
14. We must now consider a proposition, known as the *Principle of Huygens.*

Let PQ be the front, at time t, of a wave of any form which is travelling outwards; and let $P'Q'$ be the front of the wave at time t'. To fix our ideas we may suppose that the wave is spherical, and the medium isotropic, but the argument will apply to waves of any form, which are propagated in an æolotropic medium.

At time t, the ether in the neighbourhood of P will be in a state of vibration; hence P may be regarded as a centre of disturb-

[1] According to the definition adopted, the directions of propagation and vibration are the same as those of translation and rotation of a right-handed screw; they are also related in the same manner, as the magnetic force produced by an electric current circulating round the ray.

ance, which propagates a spherical wave. Draw OPP' to meet the front of the wave at time t'; then at the end of an interval $t'-t$ after the wave has passed over P, the secondary wave

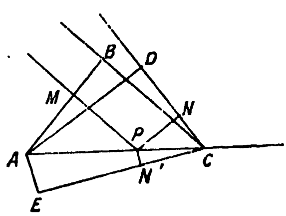


produced by the element P will have reached P', and the ether in the neighbourhood of P' will be thrown into vibration. Similarly if Q be any other point on the wave front at time t, and OQQ' be drawn to meet the wave front at time t', it follows that at time t' the ether at Q' will be set in motion, owing to the secondary wave propagated by Q. We therefore see, that the wave front at time t' is the envelop of the secondary waves, which may be conceived to diverge from the different points of the wave front at time t.

We are thus led to the following proposition, which was first enunciated by Huygens, and which may be stated as follows:—

The effect of a primary wave upon any given point in the region beyond the wave, may be obtained by dividing the primary wave into an indefinite number of small elements, which are to be regarded as centres of disturbance, and finding by integration over the front of the primary wave, the sum of the disturbances produced at the given point by each of the secondary waves.

15. It is a well-known law of Geometrical Optics, that when light is reflected at a given surface, the angle of incidence is equal to the angle of reflection. We shall now prove this law by means of the undulatory theory.



Let AB be any portion of the front of a plane wave, which is reflected at AC, M any point upon it; and draw MP, BC perpendicular to the wave front. Draw AD, CD such that the

angles DAC and DCA are respectively equal to the angles BCA and BAC; and draw PN perpendicular to CD. Let t, T be the times which the wave occupies in travelling to P and C.

Then $$PM = Vt, \quad BC = VT,$$

also since the triangles ABC and CDA are equal in every respect, and the angle D is therefore a right angle,

$$\frac{PN}{AD} = \frac{PC}{AC} = \frac{CB - PM}{CB};$$

hence since $AD = CB$,

$$PN = CB - PM = V(T - t).$$

Now when the original wave reaches P, this point will become a centre of disturbance, and spherical waves will be propagated; at the end of an interval $T - t$ the wave has reached C, and the secondary wave which diverges from P has reached N, since we have shown that $PN = V(T - t)$. Hence all the secondary waves which diverge from points between A and C will touch AC, which is accordingly the front of the reflected wave at the instant the incident wave has reached C.

Since we have shown that the triangles ABC and CDA are equal in every respect, the angle $BCA = DAC$; whence the angle of incidence is equal to the angle of reflection.

16. In order to prove the law of refraction, let V' be the velocity of light in the second medium; and with A as centre describe a sphere of radius $V'T$, and let CE be the tangent drawn from C to this sphere; draw PN' perpendicular to CE. Then

$$\frac{PN'}{AE} = \frac{PC}{AC} = \frac{CB - PM}{CB} = \frac{T - t}{T},$$

whence $$PN' = V'(T - t);$$

which shows when the incident wave has reached C, the secondary wave which diverges from P in the second medium, will have reached N'. Hence CE is the front of the refracted wave.

If i, r be the angles of incidence and refraction,

$$AC \sin i = CB = VT$$

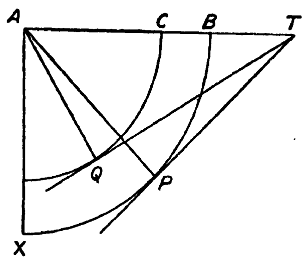
and $$AC \sin r = AE = V'T,$$

whence $$\frac{\sin i}{\sin r} = \frac{V}{V'},$$

which is the law of sines.

When the second medium is more highly refracting than the first, $i > r$, whence $V > V'$; accordingly the velocity of light is greater in a less refracting medium such as air, than in a more highly refracting medium such as glass. It also follows that the index of refraction is equal to the ratio of the velocities of light in the two media.

The direction of the refracted ray may be found by the following geometrical construction, which as we shall hereafter show may be generalized in the case of crystalline media, in which the wave surface is not spherical.



Let AB be the surface of separation of the two media, A the point of incidence. With A as a centre describe two spheres whose radii AB, AC are proportional to the velocities V, V' of light in the two media. Let the incident ray AP be produced to meet the first sphere in P, and let the tangent at P meet the plane AB in T. Then if TQ be the tangent from T to the second sphere, AQ is the refracted ray.

Since the angle

$$PTA = PAX, \text{ and } QTA = QAX,$$

$$\frac{V}{V'} = \frac{AP}{AQ} = \frac{\sin PAX}{\sin QAX}.$$

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But $PAX = i$, whence $QAX = r$.

CHAPTER II.

INTERFERENCE.

17. WE stated in the preceding Chapter, that we shall assume that the sensation of light is produced by the vibrations of the ether, without enquiring for the present into the physical constitution of the latter. We simply suppose that a medium exists which is capable of propagating waves, and that when the waves are plane, the direction of vibration is parallel to the wave front. We shall now proceed to examine how far this hypothesis is capable of explaining the interference of light.

We have shown in § 12, that the superposition of two waves of light whose directions, wave-lengths, velocities of propagation and planes of polarization are the same, but whose amplitudes and phases are different, may either intensify or diminish the resultant light. Let us now suppose, that natural light is proceeding from two sources very close to one another. At a point whose distance from the two sources is large in comparison with the distance between them, the waves may be regarded as approximately plane and parallel to one another, and the displacements may be resolved into two components at right angles to one another in the front of the wave. Hence if v, v' be any two components whose directions are parallel, we may take

$$v = A \cos \frac{2\pi}{\lambda}(x - Vt), \quad v' = A' \cos \frac{2\pi}{\lambda}(x - Vt - e),$$

the origin being suitably chosen; accordingly the resultant of these two vibrations is

$$v + v' = \mathfrak{A} \cos \frac{2\pi}{\lambda} (x - Vt - \delta) \text{ .... } (1),$$

where $$\mathfrak{A}^2 = A^2 + A'^2 + 2AA' \cos 2\pi e/\lambda \text{..............} (2),$$

and $$\tan 2\pi\delta/\lambda = \frac{A' \sin 2\pi e/\lambda}{A + A' \cos 2\pi e/\lambda} \text{..........................} (3).$$

If the amplitudes of the two waves are equal, then (2) and (3) become

$$\mathfrak{A} = 2A \cos \pi e/\lambda, \quad \delta = \tfrac{1}{2}e \text{ ....} (4),$$

accordingly the intensity of the resultant light will be zero when $e = (n + \frac{1}{2})\lambda$, and a maximum when $e = n\lambda$. It therefore follows that when the phases of the two waves differ by an odd multiple of half a wave-length, the superposition of the two waves produces darkness, and the waves are said to interfere.

In order to produce interference, it is essential that the two sources of light should arise from a common origin, otherwise it would be impossible to insure that the amplitudes of the two waves should be equal, or that the difference of phase should remain invariable; accordingly if the light from two *different* candles were made to pass through two pin-holes in a card, which are very close together, interference would not take place[1]; but if light from a single candle were passed through a pin-hole, and the resulting light were then passed through the two pin-holes, interference would take place if the two pin-holes were sufficiently close together.

The phenomenon of interference was regarded as a crucial test of the truth of the Undulatory Theory, before that theory was so firmly established as it is at present; inasmuch as it is impossible to satisfactorily explain on the Corpuscular Theory, how two lights can produce darkness.

18. We shall now explain several methods, due to Fresnel, by means of which interference fringes can be produced.

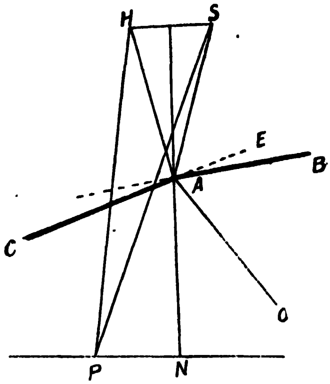
Let AB, AC be two mirrors[2] inclined at an angle $\pi - \alpha$, where α is very small; let O be a luminous point[3], and let S, H be the images of O formed by the mirrors AB, AC; also let P be any point on a screen PN, which is parallel to the line of intersection of the mirrors, and to SH.

[1] The effect of diffraction is not considered.

[2] *Œuvres Complètes*, vol. I. pp. 159, 186, 268.

[3] In practice the source usually consists of light admitted through a narrow slit.

Since $AO = AS = AH$, the difference of phase of the two



streams of light which come from S and H is $SP - HP$. Let $AO = a$, $NP = x$, d the distance of the screen from SH. Then

$$HAE = HAS + SAB - \alpha,$$

whence $$HAS = OAE - OAB + \alpha = 2\alpha;$$

accordingly $$SH = 2a \sin \alpha.$$

Now $$SP^2 = d^2 + (x + a \sin \alpha)^2,$$

$$HP^2 = d^2 + (x - a \sin \alpha)^2,$$

and since the distance of the screen is large compared with x and $a \sin \alpha$, we obtain

$$SP - HP = 2ax \sin \alpha \,. /d = SH \,. x/d.$$

Since the two images are produced by the same source, their amplitudes will be equal, whence the intensity is equal to

$$4A^2 \cos^2 \frac{2\pi ax \sin \alpha}{d\lambda} \quad \text{.......................(5).}$$

Since α is a very small angle, we may write α for $\sin \alpha$, and from (5) we see that the intensity is a maximum when

$$x = \tfrac{1}{2} n \lambda d / a\alpha,$$

and a minimum when $x = \frac{1}{4}(2n + 1) \lambda d/a\alpha$, where n is an integer.

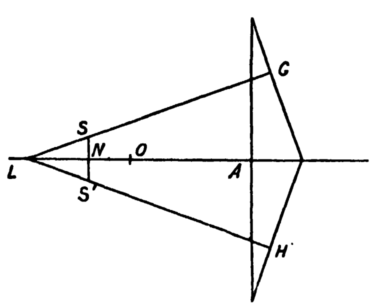
If homogeneous light is employed, a bright band will be observed at the centre N of the screen, and on either side of this bright band and at a distance $\frac{1}{4}\lambda d/a\alpha$, there will be two dark bands; accordingly the screen will be covered by a series of bright and dark bands succeeding one another in regular order; the distance between two bright bands or two dark bands being equal

to $\frac{1}{2}\lambda d/a\alpha$. From this result we see the necessity of α being very small, for otherwise on account of the smallness of λ, the bands would be so close together as to be incapable of being observed.

If perfectly homogeneous light could be obtained, the number of bands would be theoretically unlimited, and with light from a sodium flame, which possesses a high degree of homogeneity, Michelson has observed as many as 200000 bands, but in practice it is not possible to obtain absolutely homogeneous light, consequently the number of fringes is necessarily limited.

Since the distance between two bands is equal to $\frac{1}{2}\lambda d/a\alpha$, it follows that the breadths of the bands depend upon the wave-length, and therefore upon the colour; hence if sunlight be employed, a certain number of brilliantly coloured bands will be observed. At the centre of the system, where $x = 0$, the difference of phase of waves of all lengths is zero, and the central band is therefore white, but its edges are red. The inner edge of the next bright band will be violet, and its edge of a reddish colour; but as we proceed from the centre, the maximum intensity of one colour will coincide with the minimum of another, and the dark bands will altogether disappear, and will be replaced by coloured bands. At a still further distance, the colours will become mixed to such an extent, that no bands will be distinguishable.

19. In Fresnel's second experiment[1], the light was refracted by means of a prism having a very obtuse angle, which is called a biprism.



Let O be the luminous point, and let L be the focus of the rays refracted at the first face; let S and S' be the foci of the rays refracted at the second faces, and draw LG, LH perpendicular to

[1] *Œuvres Complètes*, vol. I. p. 330.

these faces. Let $OA = a$, and let μ be the index of refraction. Then

$$LA = \mu a, \quad LG = \mu SG.$$

Hence if N be the point where SS' cuts AO,

$$SN = LS \sin \alpha = LG(1 - \mu^{-1}) \sin \alpha$$
$$= (T + \mu a)(1 - \mu^{-1}) \sin \alpha \cos \alpha,$$

where T is the thickness of the prism at A.

The light may now be conceived to diverge from the two points S and S', and therefore if d be the distance of the screen from SS', the retardation will be equal to

$$\frac{1}{\mu d}(T + \mu a)(\mu - 1)\, x \sin 2\alpha,$$

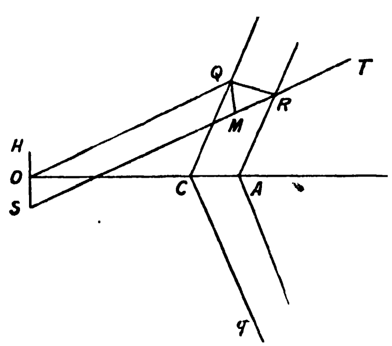
which is of the same form as in the last experiment.

The breadths of the bands are accordingly equal to

$$\frac{\mu \lambda d}{(T + \mu a)(\mu - 1) \sin 2\alpha},$$

which involves μ as well as λ. Now the index of refraction depends not only upon the material of which the prism is made, but also on the colour of the light; hence the breadths of the bands are affected by the dispersive power of the prism, but are otherwise the same as in the last experiment.

20. A third mode of producing interference fringes is by means of bi-plates.



Let CQ, Cq be two thin plates of thickness T inclined at a very obtuse angle $\pi - 2\alpha$. If O be a luminous origin situated on the axis CA of the bi-plates, a small pencil of light after passing through the plates CQ, Cq will appear to diverge from two foci S, H, such that SH is perpendicular to OC.

Let $OQRT$ be any ray, draw QM parallel to SH; then if i, r be the angles of incidence and refraction

$$QMR = \tfrac{1}{2}\pi - i + \alpha, \quad QRM = i - r,$$

whence
$$\frac{OS}{T \sec r} = \frac{\sin (i - r)}{\cos (i - \alpha)};$$

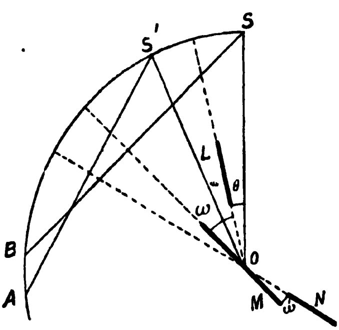
therefore
$$SH = \frac{2T}{\cos (i - \alpha)} (\sin i - \tan r \cos i).$$

Since $i = \alpha$ very nearly, and α is small, this may be approximately written

$$SH = 2T\alpha (1 - \mu^{-1}).$$

Having obtained the value of SH, the calculation proceeds as before.

147 **21.** A fourth method, which was also employed by Fresnel[1], consists of three mirrors, L, M, N, placed so that L and N intersect at a point O on M.



The light proceeds from a source S, and is reflected at the first mirror L, and is then reflected from the third mirror N. After reflection from N, the light will appear to diverge from a focus A such that

$$S'A = 2 (\omega' + \omega - \theta),$$

where θ is the inclination of SO to L; and ω, ω' are the angles which L and N make with M.

The light reflected at M appears to diverge from a focus B, such that

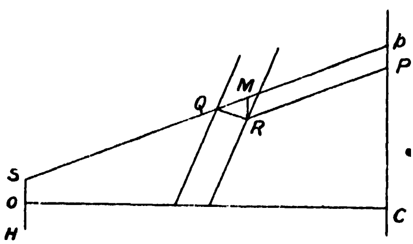
$$SB = 2 (\omega + \theta),$$

whence
$$AB = SA - SB = 2\theta + S'A - SB$$
$$= 2 (\omega' - \theta).$$

[1] *Œuvres Complètes*, vol. I. p. 703.

Hence if $\omega' - \theta$ is small, the distance AB will be small, and the two pencils proceeding from A, B will be in a condition to interfere.

22. When interference fringes are viewed through a prism, or through a plate of glass held obliquely to the screen, the fringes will be displaced, and we shall now calculate the displacement.



Let T be the thickness of the plate, μ its index of refraction, β its inclination to the screen; also let $SQRP$ be any ray, and draw RM parallel to CP. Let R_1 be the retardation, and let $CP = x$, $OS = c$; then

$$R_1 = SQ + \mu QR + RP.$$

In calculating R_1, we shall consider β, and i the angle of incidence, to be small quantities, and we shall neglect cubes and higher powers.

Now $$\frac{RM}{QR} = \frac{\sin(i - r)}{\cos(i - \beta)},$$

whence $$RM = \frac{T}{\mu}(\mu - 1)\, i.$$

Also $$\frac{QM}{QR} = \frac{\cos(\beta - r)}{\cos(i - \beta)},$$

whence $$QM = \frac{T\cos(\beta - r)}{\cos r \cos(i - \beta)}.$$

Now $$d^2 + (x - c + RM)^2 = Sp^2 = (SQ + QM + RP)^2,$$

whence

$$SQ + RP = \left[d^2 + \left\{x - c + \frac{T}{\mu}(\mu - 1)\, i\right\}^2\right]^{\frac{1}{2}} - \frac{T\cos(\beta - r)}{\cos r \cos(i - \beta)}$$

$$= d + \frac{1}{2\mu^2 d}\{\mu(x - c) + T(\mu - 1)\, i\}^2$$

$$- T + \tfrac{1}{2}T\{(\beta - r)^2 - r^2 - (i - \beta)^2\},$$

whence $$R_1 = d + \frac{1}{2\mu^2 d}\{\mu(x-c) + T(\mu-1)i\}^2$$
$$+\tfrac{1}{2}T(\mu-1)\left(2 - \frac{i^2}{\mu} + \frac{2\beta i}{\mu}\right).$$

We must now find i in terms of x. We have
$$x + RM - c = d\tan(i-\beta),$$
or $$\mu(x-c) + T(\mu-1)i = \mu d(i-\beta),$$
whence $$i = \frac{\mu\beta d + \mu(x-c)}{\mu d - T(\mu-1)} = \beta$$
approximately, accordingly
$$R_1 = d + \frac{1}{2\mu^2 d}\{\mu(x-c) + T(\mu-1)\beta\}^2 + \tfrac{1}{2}T(\mu-1)\left(2 - \frac{\beta^2}{\mu}\right).$$

The value of R_2, the retardation of a ray proceeding from H to P, is obtained by changing c into $-c$; whence
$$R_2 - R_1 = \frac{2cx}{d} + \frac{2(\mu-1)Tc\beta}{\mu d} = \delta.$$

The original central band was $x=0$, and the central band which is determined by $\delta = 0$ is now given by
$$x = -\frac{(\mu-1)T\beta}{\mu} \quad \text{.....................(6)},$$
which shows that it is shifted through a distance $-T(1-\mu^{-1})\beta$.

23. When interference fringes are examined through a prism, the displacement of the central band is different from the theoretical result given by (6). This difficulty was first explained by Airy[1], who drew attention to the fact that when no prism is used, the central band is the locus of the points *for which all colours of the light composing the two pencils have travelled over equal paths.* Now from (6) it appears, that the displacement of the points which formerly constituted the central band, depends upon μ the index of refraction of the prism; and this quantity is different for different colours, being greatest for violet and least for red light. Since the original central band consists of a mixture of light of every colour, it follows from (6) that the displacement of the red portion of the band will be less than that of the violet, and consequently the portion of the central band which is nearest C will be red, whilst the farthest portion will be violet. This band

[1] *Phil. Mag.* 1833, p. 161.

can therefore no longer be considered the central or *achromatic* band.

The actual achromatic band is determined from the consideration, that if the bands of all colours coincide at any particular part of the spectrum, they will coincide at no other part; hence if v be the displacement, measured in the direction CP, of the original central band, the distance x_n of the nth band after displacement will be

$$x_n = v + x\lambda d/2c.$$

The achromatic band occurs when x_n is as nearly as possible independent of λ, that is when $dx_n/d\lambda = 0$, in which case n must be the integer nearest to

$$-\frac{2c}{d}\frac{dv}{d\lambda}.$$

Since the width h of a band is equal to $\lambda d/2c$, this may be written

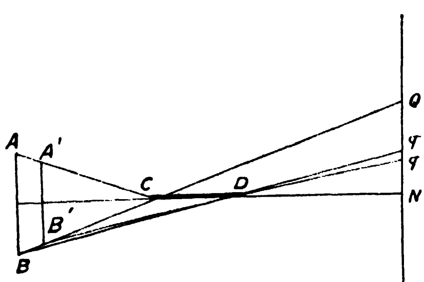
$$n = -\frac{dv}{dh},$$

so that the apparent displacement of the achromatic band is[1]

$$v - h\frac{dv}{dh}.$$

24. The following method of producing interference fringes was devised by Lloyd[2].

A luminous point A is reflected from a plane mirror CD at nearly grazing incidence. The reflected rays accordingly emerge



from a virtual focus B, and the arrangement is therefore equivalent to two small sources of light very close together. Let BC,

[1] See also Cornu, *Jour. de Phys.* vol. I. p. 293 (1882). Lord Rayleigh, "On achromatic interference bands," *Phil. Mag.* (5), vol. XXVIII. pp. 77 and 189.

[2] *Trans. Roy. Ir. Acad.* vol. XVII.

BD meet the screen in *Q*, *q*; then since interference is due to the mixture of the two streams of light, the bands will only exist between the points *Q*, *q*. Moreover since the difference of path is never zero, there can be no achromatic band.

The achromatic band may however be rendered visible by placing a thin plate of glass in the path of the direct pencil. Putting $AB = 2c$ the retardation at x is

$$2cx/d - T(\mu - 1),$$

whence

$$x_0 = T(\mu - 1)\, d/2c,$$

and
$$x_n = T(\mu - 1)\, d/2c + n\lambda d/2c,$$

and consequently the position of the achromatic band, which is determined by $dx_n/d\lambda = 0$, will be given by n, where n is the integer nearest to

$$-\, Td\mu/d\lambda.$$

One peculiarity must be noticed, and that is that the band, which corresponds to a zero difference of path, is not white but black. Now when we consider the dynamical theory of reflection and refraction, it will be found that at grazing incidence, the amplitude of the reflected light is very nearly equal to that of the incident light, but is negative. From (2) we see that when $A' = -A$, the intensity of the mixture is proportional to $4A^2 \sin^2 \pi e/\lambda$, which vanishes when $e = 0$. The adjoining bright band is given by $e = \frac{1}{2}\lambda$, or

$$x = T(\mu - 1)\, d/2c + \lambda d/4c.$$

EXAMPLES.

1. A small pencil of light is reflected at three mirrors, so that the images form a small triangle *ABC*, of which *C* is a right angle. Prove that the intensity at any point (x, y) on a parallel screen at a distance d, is proportional to

$$1 + 8 \cos \frac{\pi}{\lambda d}(ax - cy) \cos \frac{\pi a x}{\lambda d} \cos \frac{\pi c y}{\lambda d};$$

where $AC = a$, $BC = c$; and the projection of C on the screen is the origin, and CA is the axis of x.

2. A small source of homogeneous light is reflected in three mirrors, in such a manner that the images are equally bright and form an equilateral triangle *abc*, whose centre of gravity is *o*. *A*, *B*, *C*, *O* are the projections of *a*, *b*, *c*, *o* upon a screen which is parallel to the plane *a*, *b*, *c*. Show that the intensity at any point *P* in the line *OA*, is to that at *O*, in the ratio

$$1 - \tfrac{8}{9}\sin^2 3\pi\rho d/2\lambda h : 1,$$

where h is the distance of the screen, and $PO = \rho$, $oa = d$.

CHAPTER III.

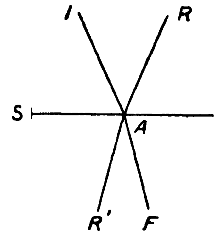
COLOURS OF THIN AND THICK PLATES.

25. WHEN light is incident upon a thin film of a transparent substance, such as a soap-bubble, brilliant colours are observed. The explanation of this phenomenon is, that the light upon incidence upon the outer surface of the film, is separated into two portions, the first of which is reflected by the outer surface, whilst the second portion is refracted. The refracted portion is reflected from the second surface of the film, and afterwards refracted by the outer surface; and since the thickness of the film is very small, the difference of the paths of the two portions is comparable with the wave-length, and the two streams are therefore in a condition to interfere. Accordingly if sunlight is employed, a series of brilliantly coloured bands is observed.

In order to obtain a mathematical theory of these bands, we shall suppose that two plates of glass cut from the same piece, are placed parallel to one another with a thin stratum of air between them; and we shall investigate the intensity of the reflected light.

26. It will be proved in a future Chapter, that when light is reflected or refracted at the surface of a transparent medium, the intensities of the reflected and refracted light are altered in a manner, which depends upon the angle of incidence and the index of refraction. The mathematical formulæ, which determine the intensities in these two cases, depend partly upon the particular dynamical theory which we adopt, and partly upon the state of polarization of the light. If however the angle of incidence of the light, which is refracted from the plate into the stratum of air, is less than the critical angle, we can achieve our object

without the assistance of any dynamical theories, by the aid of a principle due to Stokes[1], called the Principle of Reversion.



Let S be the surface of separation of two uncrystallized media; let A be the point of incidence of a ray travelling along IA in the first medium, and let AR, AF be the reflected and refracted rays; also let AR' be the direction of the reflected ray for a ray incident along FA in the second medium. Then the principle asserts, that if the two rays AR, AF be reversed, so that RA and FA are reflected and refracted at A, they will give rise to the incident ray AI.

Let A be the amplitude of the incident light; and let Ab, Ac be the amplitudes of the reflected and refracted light, when the first medium is glass and the second is air; also let Ae, Af be the amplitudes of the reflected and refracted light, when light of amplitude A is refracted from air into glass. Then if AR be reversed, it will give rise to

$$Ab^2 \text{ reflected along } AI,$$
$$Abc \text{ refracted along } AR'.$$

Similarly if AF be reversed, it will give rise to

$$Ace \text{ reflected along } AR',$$
$$Acf \text{ refracted along } AI.$$

Since the two rays superposed along AR' must destroy one another, whilst the two rays superposed along AI must be equivalent to the incident ray, we obtain

$$b+e=0,\ b^2+cf=1 \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

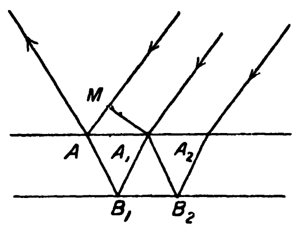
27. We are now in a position to calculate the intensity.

Let $$y = A \sin 2\pi t/\tau$$

be the incident vibration at A; let i be the angle of incidence, r that of refraction, D the thickness of the stratum of air.

[1] *Camb. and Dublin Math. Journ.* vol. IV. p. 1; and *Math. and Phys. Papers*, vol. II. p. 89.

The light which is incident at A_1 is reflected and refracted, and a portion of the latter is reflected at B_1, and the reflected



portion is again reflected and refracted at A, and so on ad infinitum. It therefore follows, that the light refracted at A which is due to light incident at A_1, is represented by

$$Acef \sin 2\pi \left(\frac{t}{\tau} + \frac{2AB_1}{\lambda}\right) \dots\dots\dots\dots\dots\dots (2).$$

Also if λ' be the wave-length in glass, the vibration at A due to the light which is reflected at A_1 is

$$Ab \sin 2\pi \left(\frac{t}{\tau} + \frac{AM}{\lambda'}\right).$$

Writing

$$\phi = 2\pi \left(\frac{t}{\tau} + \frac{AM}{\lambda'}\right), \quad \delta = 2\pi \left(\frac{2AB}{\lambda} - \frac{AM}{\lambda'}\right) \dots\dots (3),$$

(2) may be written $\qquad Acef \sin(\phi + \delta)$,

where δ is the retardation of the light which was refracted at A_1. Taking account of (1), and also of the infinite series of reflections and refractions at A_2, A_3 ...... etc., we obtain for the resulting vibration at A

$$y = Ab\,[\sin\phi - (1 - b^2)\,\{\sin(\phi + \delta) + b^2 \sin(\phi + 2\delta) + b^4 \sin(\phi + 3\delta) + \dots\dots\}].$$

Summing this series we obtain

$$y = \frac{2Ab\,(1 + b^2)\sin^2 \tfrac{1}{2}\delta \sin\phi - Ab\,(1 - b^2)\sin\delta\cos\phi}{1 - 2b^2\cos\delta + b^4},$$

whence the intensity is equal to

$$I^2 = \frac{4A^2b^2 \sin^2 \tfrac{1}{2}\delta}{(1 - b^2)^2 + 4b^2 \sin^2 \tfrac{1}{2}\delta} \dots\dots\dots\dots\dots\dots (4).$$

Since $AB_1 = D \sec r$, $AM = 2D \tan r \sin i$, $\lambda'/\lambda = \sin i/\sin r$, we obtain

$$\delta = 4\pi\lambda^{-1} D \cos r \dots\dots\dots\dots\dots\dots\dots\dots (5).$$

Although we have supposed at the commencement, that a thin film of air is contained between two plates of glass very near one another, it is evident that the preceding investigation will apply equally well to a soap-bubble.

If $D=0$, then $\delta=0$ and $I=0$; accordingly when a soap-bubble becomes exceedingly thin just before bursting, it appears to be black.

The intensity will also vanish when $2D\cos r = n\lambda$, where n is an integer; but since this condition depends upon λ, it follows that when sunlight is employed, the intensity will never become absolutely zero, but the film will be coloured.

In order to obtain the intensity of the transmitted light, it can be shown in a precisely similar manner, that the vibration which emerges at B_1 is represented by

$$y = Acf\{\sin\phi + e^2\sin(\phi+\delta) + e^4\sin(\phi+2\delta) + \dots\dots\}.$$

Summing this series, and taking account of (1), we shall find that the intensity is

$$I_1^2 = \frac{A^2(1-b^2)^2}{(1-b^2)^2 + 4b^2\sin^2\frac{1}{2}\delta} \quad \dots\dots\dots\dots\dots(6).$$

Adding (4) and (5) we see that

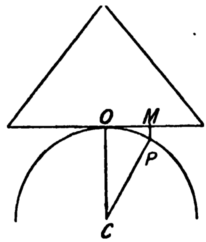
$$I^2 + I_1^2 = A^2 \quad \dots\dots\dots\dots\dots\dots\dots\dots(7)$$

or the sum of the intensities of the reflected and transmitted lights is equal to that of the incident light. This result is sometimes expressed by saying, that the reflected and transmitted lights are complementary to one another. It must however be borne in mind, that (7) is not strictly accurate for ordinary transparent media, inasmuch as a portion of the light is always absorbed in transmission through the plate; it only becomes true in the limit for perfectly transparent substances.

Newton's Rings.

28. The coloured rings produced by thin plates were first investigated experimentally by Newton, who produced them by pressing a convex lens down upon a flat piece of glass; the experiment may also be performed by pressing a prism upon the face of a convex lens. Since the curvature of the lens is exceedingly small in comparison with the wave-length of light, the two surfaces

may be regarded as approximately parallel, and the preceding investigation will apply.



Let R be the radius of the lens, O the point of contact, and let $OM=\rho$. Then $PM=D$, and

$$(2R-D)\,D=\rho^2,$$

whence neglecting D^2, we have $D=\rho^2/2R$, accordingly

$$\delta=2\pi\rho^2/R\lambda\,.\cos r \quad\ldots\ldots\ldots\ldots\ldots\ldots (8),$$

and the reflected light vanishes when

$$\delta=2n\pi \text{ or } \rho^2=nR\lambda\sec r \quad\ldots\ldots\ldots\ldots\ldots\ldots (9).$$

At O, $\rho=0$ and therefore $\delta=0$; whence the central spot is black. If homogeneous light is employed, the central spot will be surrounded by a series of dark rings, whose diameters are proportional to the square roots of the natural numbers.

The intensity will be a maximum, when

$$\delta=(2n+1)\,\pi \text{ or } \rho^2=(n+\tfrac{1}{2})\,R\lambda\sec r \quad\ldots\ldots\ldots\ldots (10);$$

accordingly there will be a series of bright rings whose diameters are proportional to $\sqrt{\tfrac{1}{2}}$, $\sqrt{\tfrac{3}{2}}$, $\sqrt{\tfrac{5}{2}}$ etc.

Since the diameters of the rings are also proportional to $(\sec r)^{\frac{1}{2}}$, it follows that the rings increase as the angle of incidence increases.

For light of different colours, the diameters of the rings vary as $\lambda^{\frac{1}{2}}$; consequently when sunlight is employed, a number of coloured rings are observed.

The inner edge of the first ring is dark blue, and its outer edge red; and the order of succession of the colours of the first seven rings was found by Newton to be as follows:—(1) black, blue, white, yellow, red; (2) violet, blue, green, yellow, red; (3) purple, blue, green, yellow, red; (4) green, red; (5) greenish-blue, red; (6) greenish-blue, pale red; (7) greenish-blue, reddish-white. This

list is usually known as *Newton's scale of colours;* and the expression "red or blue of the third order," refers to the colour of that name seen in the third ring.

In the preceding discussion of Newton's rings, we have supposed that a thin stratum of air constitutes the thin plate; consequently it is possible to increase the angle of incidence until it exceeds the critical angle. Under these circumstances, it will be found that there is still a system of coloured rings, and that the central spot is black; but the consideration of this question must be deferred, until we have discussed the dynamical theory of reflection and refraction.

The system of transmitted rings is complementary to the reflected system, but is less distinct.

Colours of Thick Plates.

29. The phenomenon known as the *colours of thick plates* was first observed by Newton[1], who allowed sunlight, proceeding into a darkened room through a hole in the window-shutter, to fall perpendicularly upon a concave mirror formed of glass quicksilvered at the back. A white opaque card pierced with a small hole was placed at the centre of curvature of the mirror, so that the regularly reflected light returned through the small hole, and a set of coloured rings was observed on the card surrounding the hole. The Duc de Chaulnes[2] on repeating this experiment, observed that the brilliancy of the rings was much increased by spreading over the surface of the mirror a mixture of milk and water, which was allowed to dry, and thus produced a permanent tarnish. The colours of thick plates were first explained on the undulatory theory by Young, who attributed them to the interference of two streams of light, one of which is scattered on entering the glass, and then regularly reflected and refracted, whilst the other is regularly reflected and refracted, and then scattered on emerging from the first surface; but the complete explanation is due to Stokes[3], which we shall now consider.

[1] *Optics*, Book II. part 4.

[2] *Mém. de l'Académie*, 1755, p. 136.

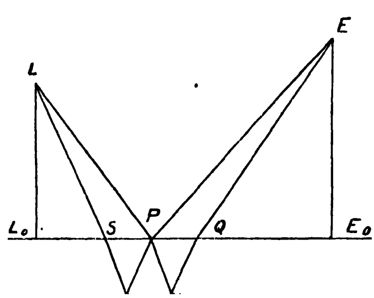
[3] "On the Colours of Thick Plates," *Trans. Camb. Phil. Soc.* vol. IX. p. 147.

30. Stokes' investigation is based on the following hypothesis:—

In order that two streams of scattered light may be capable of interfering, it is necessary that they should be scattered, in passing and repassing, by the same set of particles. Two streams, which are scattered by different sets of particles, although they may have come originally from the same source, behave with respect to each other like two streams coming from different sources.

It will hereafter be proved, that if this law were not true, it would follow, that if a luminous point were viewed through a plate of glass, *both* of whose surfaces were tarnished with milk and water, coloured rings would be seen; but on performing the experiment no rings were observed. Moreover Stokes calculated the retardation of the stream scattered on emergence relatively to that scattered at entrance, and found that the dimensions of the rings were such that they could not possibly have escaped notice had they been formed. This experiment is decisive, but the truth of the law is also apparent from theoretical considerations; for the dimensions of particles of dust, although small compared with the standards of ordinary measurement, are not small in comparison with the wave-length of light, so that the light scattered at entrance taken as a whole is most irregular; and the only reason why regular interference is possible at all is, that each particle acts twice in a similar manner, once when the wave enters and again when it emerges.

31. We shall now work out the problem when the mirror is plane.



Let L be the luminous point, E the eye of the observer; let L_0, E_0 be the feet of the perpendiculars let fall from L and E on to the dimmed face of the mirror.

Let $LSTPE$ be the course of the ray, which is regularly refracted and reflected at S and T, and scattered on emergence at P; and let $LPVQE$ be the course of the ray, which is scattered entering the glass and is then regularly reflected and refracted.

Let $L_0P = s$, $E_0P = u$, $LL_0 = c$, $EE_0 = h$; also let t be the thickness of the plate, μ its index of refraction, i, r the angles of incidence and refraction at S; R_1, R_2 the retardations of the rays $LSTPE$ and $LPVQE$. Then

$$R_1 = LS + 2\mu ST + PE$$

$$= c \sec i + 2\mu t \sec r + (h^2 + u^2)^{\frac{1}{2}} \ldots\ldots (11),$$

and $$c \tan i + 2t \tan r = s, \quad \sin i = \mu \sin r \ldots\ldots\ldots\ldots (12).$$

Now experiment shows, that in order to see the rings distinctly the angle of incidence must be small, whence i, r, s and u are small quantities. We may therefore, as a sufficient approximation, neglect powers of small quantities above the second. Expanding in powers of i, r and u, we obtain from (11)

$$R_1 = c + 2\mu t + h + \tfrac{1}{2}(ci^2 + 2\mu t r^2 + u^2/h) \ldots\ldots\ldots\ldots (13).$$

But (12) gives $$i = \mu r = \frac{\mu s}{\mu c + 2t},$$

whence $$R_1 = c + 2\mu t + h + \frac{\mu s^2}{2(\mu c + 2t)} + \frac{u^2}{2h} \ldots\ldots\ldots\ldots (14).$$

Again, if i', r' be the angles of incidence and refraction at Q,

$$R_2 = LP + 2\mu PV + QE$$

$$= (c^2 + s^2)^{\frac{1}{2}} + 2\mu t \sec r' + c \sec i',$$

and $$h \tan i' + 2t \tan r' = u, \quad \sin i' = \mu \sin r'.$$

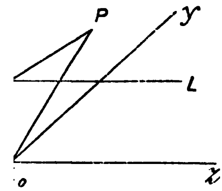
Accordingly we obtain as before

$$R_2 = h + 2\mu t + c + \frac{\mu u^2}{2(\mu h + 2t)} + \frac{s^2}{2h} \ldots\ldots\ldots\ldots (15),$$

whence $$R = R_1 - R_2 = t\left\{\frac{u^2}{h(\mu h + 2t)} - \frac{s^2}{c(\mu c + 2t)}\right\} \ldots\ldots\ldots (16).$$

The intensity of the light entering the eye is therefore proportional to $\cos^2 \pi R/\lambda$.

Let E_0 be the origin, and let E_0E be the axis of z, and let the plane xz pass through L. Let x, y be the coordinates of P, and let $E_0L_0 = a$.



Then
$$\left.\begin{aligned} u^2 &= E_0P^2 = x^2 + y^2 \\ s^2 &= L_0P^2 = (a-x)^2 + y^2 \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots\ldots(17).$$

Also let the thickness of the plate be supposed to be so small, that its square may be neglected. Then substituting in (16), we obtain

$$R = \frac{t}{\mu}\left\{\left(\frac{1}{h^2} - \frac{1}{c^2}\right)(x^2+y^2) + \frac{2ax}{c^2} - \frac{a^2}{c^2}\right\} \ldots\ldots\ldots\ldots(18).$$

For a given fringe R is constant; hence the fringes form a system of concentric circles, whose common centre lies on the axis of x. Hence if α be the abscissa of the centre

$$\alpha = \frac{ah^2}{h^2 - c^2} = \tfrac{1}{2}\left(\frac{ah}{h+c} + \frac{ah}{h-c}\right) \ldots\ldots\ldots\ldots\ldots(19).$$

Now $ah/(h+c)$ and $ah/(h-c)$ are the abscissæ of the points, in which the plane of the mirror is cut by two lines drawn from the eye to the luminous point and its image respectively. We thus obtain the following construction for finding the centre of the system:—*Join the eye with the luminous point and its image, and produce the former line to meet the mirror; then the middle point of the line joining the two points, in which the mirror is cut by the two lines joining the eye, will be the centre of the system.*

Hence if the luminous point be placed to the right of the perpendicular let fall from the eye on to the plane of the mirror, and between the mirror and the eye, the concavity of the fringes will be turned to the right. If the luminous point, still lying on the right, be now moved backwards, so as to come beside the eye and ultimately fall behind it, the curvature will decrease until the fringes become straight, after which it will increase in the

contrary direction, the convexity being now turned towards the right.

The circle $R=0$ may be called an achromatic line, since at every point of it the intensity is independent of the wave-length. It evidently passes through the two points mentioned in the last paragraph but one.

When the luminous point is situated in the line drawn through the eye perpendicularly to the mirror, $a=0$, and (18) becomes

$$R=\frac{c^2-h^2}{\mu c^2 h^2}(x^2+y^2).$$

In this case the achromatic line is reduced to a point; for the bright ring of the first order $R=\pm\lambda$, and therefore the radius of the ring is equal to

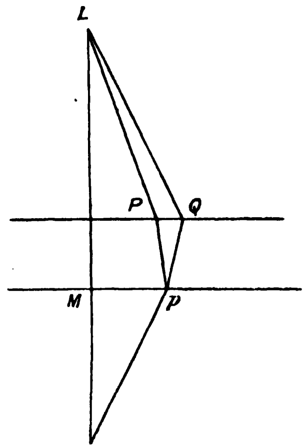
$$ch\left\{\frac{\mu\lambda}{(h^2-c^2)t}\right\}^{\frac{1}{2}},$$

which becomes infinite when $c=h$. Hence if the luminous point be at first situated in front of the eye, and be then conceived to move backwards through the eye till it passes behind it, the rings will expand indefinitely and then disappear, and will reappear again when the luminous point has passed the eye.

This result cannot be directly compared with experiment; but an analogous experiment was performed by Stokes in the following manner. Instead of a luminous point, he used the image of the sun in a small concave mirror, and placed a piece of plate-glass between the concave mirror and a plane mirror, the surface of which had been prepared with milk and water. The plate of glass was situated at a distance of some feet from the plane mirror, and was inclined at an angle of about 45°. The greater part of the light coming from the image of the sun, was transmitted through the plate of glass; and on returning from the large mirror, a portion of this light was reflected sideways, so that the rings could be seen by reflection in the plate of glass without obstructing the incident light. The system of rings thus seen was very beautiful; and Stokes found that on moving back the head, the rings expanded until the bright central patch surrounding the image filled the whole field of view, and on continuing to move back the head the rings reappeared again. In the position in which the central patch filled the whole field of view, the least motion of the eye sideways was sufficient to bring into the field of view excessively broad coloured bands.

32. We stated in § 30, that experiment showed that no rings could be produced, unless the scattering was caused by the same set of particles.

To prove this, let LE, the line joining the luminous point and the eye, be perpendicular to the plate; and let $LPpE$ be a ray



which is regularly refracted at P, and scattered at emergence at p; and let $LQpE$ be a ray which is scattered at entrance at Q, and regularly refracted at p. Then

$$\begin{aligned} R_1 &= LP + \mu Pp + Ep \\ &= c \sec i + \mu t \sec r + Ep \\ &= c + \mu t + Ep + \tfrac{1}{2}(ci^2 + \mu t r^2), \end{aligned}$$

where i, r are the angles of incidence and refraction at P. Now if $\rho = Mp$

$$\rho = c \tan i + t \tan r,$$

whence $$i = \mu r = \frac{\mu\rho}{\mu c + t}.$$

Accordingly $$R_1 = c + \mu t + Ep + \frac{\mu\rho^2}{2(\mu c + t)}.$$

Again, if i', r' be the angles of incidence and refraction at p,

$$R_2 = LQ + \mu Qp + Ep.$$

But $$LQ^2 = c^2 + (\rho + t \tan r')^2, \text{ and } \rho/h = i',$$

whence $$R_2 = c + \frac{\rho^2}{2c}\left(1 + \frac{t}{\mu h}\right)^2 + \mu t + \frac{\rho^2 t}{2\mu h^2} + Ep,$$

whence $$R_1 - R_2 = -\frac{\rho^2 t}{2\mu}\left(\frac{1}{c} + \frac{1}{h}\right)^2 \quad \ldots\ldots\ldots\ldots\ldots \quad (20).$$

If therefore it were possible for light scattered by different particles to interfere, it would follow that there would be a series of rings whose radii are determined by means of (20). No such rings are however found to exist; and Stokes has shown, by substituting numerical values in (20), that the dimensions of the rings were such that they could not possibly have escaped notice had they been formed; hence the fundamental hypothesis enunciated in § 30 is proved to be true.

Colours of Mixed Plates.

33. The colours of mixed plates were first discovered by Young[1], and are produced by interposing between two plates of glass pressed together, a mixture composed of two different materials, such as water and air; and it was found on viewing a luminous point through the plates, that a system of coloured rings was produced, which were considerably larger than the rings produced, when the intervening medium was air. Further experiments were made by Brewster[2], who employed various materials, such as transparent soap and whipped cream; but he obtained the best results by using the white of an egg beaten up into froth. To obtain a proper film of this substance, he placed a small quantity between two glass plates, and after having pressed it out into a film, he separated the glasses, and held them for a short time near the fire so as to drive off some of the superfluous moisture. The two glasses were then placed in contact and pressed together.

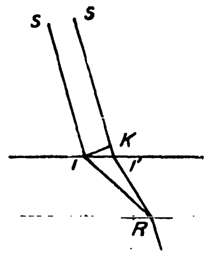
Young attributed the colours of mixed plates to the fact, that owing to the liquid being divided into an immense number of separate globules, some of the rays are transmitted through air, whilst others are transmitted through the liquid; and since the velocity of light is less in a liquid than in air, a difference of phase is produced, and thus the emergent light is in a condition to interfere.

34. The following theoretical investigation is due to Verdet[3].

Let SI, $S'I'$ be two parallel rays incident upon the surface of a mixed plate at an angle i, and let the refracted ray IR be supposed to pass through air, and the ray $I'R$ through the liquid.

[1] *Phil. Trans.* 1802. [2] *Ibid.* 1838.
[3] *Leçons d'Optique Physique*, vol. I. p. 155.

Also let μ, μ' be the indices of refraction of the glass and liquid referred to air. Then if IK be perpendicular to $I'S'$, and R_1 be the retardation $R_1 = \mu I'K + \mu' IR - IR$.



Let r, r' be the angles of refraction at I, I'; t the thickness of the space between the glass plates; then

$$IR = t \sec r, \quad I'R = t \sec r',$$

$$I'K = t (\tan r - \tan r') \sin i,$$

whence $$R_1 = t \{\mu (\tan r - \tan r') \sin i - \sec r + \mu' \sec r'\}.$$

But $$\mu = \frac{\sin r}{\sin i}, \quad \frac{\mu}{\mu'} = \frac{\sin r'}{\sin i},$$

whence $$R_1 = t (\mu' \cos r' - \cos r),$$

which may be written in the form

$$R_1 = t \{(\mu'^2 - \mu^2 \sin^2 i)^{\frac{1}{2}} - (1 - \mu'^2 \sin^2 i)^{\frac{1}{2}}\}.$$

The intensity will therefore be a maximum or minimum, according as R_1 is equal to $n\lambda$ or $(n + \frac{1}{2}) \lambda$, where λ is the wave-length in air.

At normal incidence, the intensity will be a maximum when

$$R_1 = n\lambda = t (\mu' - 1).$$

Now if Newton's rings be formed by a prism and a lens, the radii of the rings will be equal to $(2tR)^{\frac{1}{2}}$; whence if ρ be the radii of the bright rings seen by transmission when a mixture of air and liquid is employed

$$\rho^2 = \frac{2n\lambda R}{\mu' - 1}.$$

If ρ' be the radii of the bright rings seen by transmission when the thin plate is composed of air alone, it follows from (5) and (6), that at normal incidence $2D = n\lambda$,

where D is the thickness; and since $\rho'^2 = 2DR$, this becomes

$$\rho'^2 = n\lambda R,$$

whence
$$\frac{\rho^2}{\rho'^2} = \frac{2}{\mu' - 1}.$$

If the mixed plate consists of froth composed of air and water, $\mu' = \frac{4}{3}$, whence

$$\rho = \rho'\sqrt{6}.$$

The radii of the rings are therefore increased in the ratio $\sqrt{6}$ to 1. This result has been found to agree with observation.

EXAMPLES.

1. Two plates of crown and fluid glass, whose refractive indices are μ, μ', form four parallel plane reflecting and refracting surfaces. Light of wave-length λ in air could pass through each plate in the same time. A beam of parallel rays, proceeding from an origin between the two plates, and incident at an angle i on the front of the crown-glass, is partly reflected once from the front and partly once from the back, and these two reflected beams are afterwards reflected once from the back and front of the fluid glass respectively. Prove that the two beams in their parallel air courses will differ finally by

$$\frac{\mu T}{\lambda}\left(\frac{1}{\mu^2} - \frac{1}{\mu'^2}\right)\sin^2 i$$

wave-lengths, where T is the thickness of the crown-glass, and $\sin^4 i$ is to be neglected.

2. If the eye be placed in the perpendicular from a luminous point on to a dimmed plane mirror, and the thickness of the glass be small, prove that the retardation which gives rise to the rings will be

$$\left(\frac{1}{e^2} - \frac{1}{u^2}\right)\frac{Tx^2}{\mu},$$

where e and u are the distances from the eye and luminous point to the mirror, and x that of the point of scattering from the foot of the perpendicular.

3. If a plate of glass be pressed down in contact with the origin, upon a piece of glass the equation to whose bounding surface is $z = ax^2y^2$, where a is a very small quantity, describe the appearance presented by the reflected light.

CHAPTER IV.

DIFFRACTION.

35. WHEN light after passing through an aperture, whose dimensions are comparable with the ordinary standards of measurement, is received upon a screen, the boundary of the luminous area is well defined; similarly if an obstacle of sufficient size is placed in the path of the incident light, a well-defined shadow of the obstacle is cast upon the screen. It thus appears that, as long as the apertures or obstacles with which we are dealing are of moderate dimensions, light travels in straight lines. Now it is well known that sound does not in all cases travel in straight lines; for if a band is playing a piece of music out of doors, a person seated in a room with an open window can hear the music distinctly, even though his position may be such as to prevent him seeing any of the musicians. The objection was therefore raised against the undulatory theory in its infancy, that inasmuch as sound is known to be due to aerial waves, and that such waves are able to bend round corners, a theory which seeks to explain optical phenomena by means of the vibrations of a medium, ought to lead to the conclusion that light as well as sound is capable of bending round corners, which is contrary to ordinary experience. The reason of this apparent discrepancy between observation, and what was at first supposed to be the result of the undulatory theory, arises from the fact that the wave-length of light is exceedingly small compared with the linear dimensions of such apertures and obstacles as are ordinarily met with, whilst the wave-lengths of audible sounds are not small compared with them[1]. In fact it requires as extreme conditions to produce a shadow in the case of sound, as it does to avoid producing one in the case of

[1] The wave-length of the middle *c* of a pianoforte is about 4·2 feet.

light. At the same time it is quite possible for a sound-shadow to be produced. Thus:—"Some few years ago a powder-hulk exploded on the river Mersey. Just opposite the spot, there is an opening of some size in the high ground which forms the watershed between the Mersey and the Dee. The noise of the explosion was heard through this opening for many miles, and great damage was done. Places quite close to the hulk, but behind the low hills through which the opening passes, were completely protected, the noise was hardly heard, and no damage to glass and such like happened. *The opening was large compared with the wave-length of the sound*[1]."

On the other hand it is not difficult to produce a sound-shadow with an obstacle of small dimensions, by means of a sensitive flame and a tuning-fork, which yields a note whose wave-length is so short as to be inaudible; for although the vibrations of the air produced by the tuning-fork are incapable of affecting the ear, yet they are capable of producing a well-marked disturbance of the sensitive flame, by means of which the existence or non-existence of the sound is made manifest. And if an obstacle be held between the tuning-fork and the flame, it is observed that the oscillations of the latter either cease altogether or appreciably diminish, which shows that a sound-shadow has been produced[2].

36. When light passes through an aperture, such as a narrow slit, whose dimensions are comparable with the wave-length of light. and is received on a screen, it is found that a well-defined shadow of the boundary of the aperture is no longer produced. If homogeneous light be employed, a series of bright and dark bands is observed on those portions of the screen, which are quite dark when the dimensions of the aperture are large in comparison with the wave-length of light; and if white light be employed, a series of coloured bands is produced. Experiments with small apertures thus show, that light is capable of bending round corners under precisely the same conditions as sound; and thus the objections which were formerly advanced against the undulatory theory fall to the ground. These phenomena are usually known by the name of Diffraction, the object of the present

[1] Glazebrook, *Physical Optics*, p. 149.

[2] For further information on the Diffraction of Sound, see Lord Rayleigh's *Theory of Sound*, ch. XIV. and *Proc. Roy. Inst.* Jan. 20, 1888.

chapter is to show, that they are capable of being accounted for by means of the undulatory theory.

37. Let us suppose that plane waves of light are passing through an aperture in a screen, whose plane is parallel to that of the wave-fronts. Each wave upon its arrival at the aperture may be conceived to be divided into small elements dS. If O be any point at a distance from the screen, it is clear that every element dS must contribute something to the disturbance which exists at O. When we consider the dynamical theory of diffraction[1], it will be shown that, *if we suppose that the disturbance existing in that portion of the wave which passes through the aperture, is the same as if the screen in which the aperture exists were not present,* or *that the wave passed on undisturbed;* the vibration at O produced by an element dS of the primary wave, would be represented by the expression

$$\frac{cdS}{2\lambda r}(1+\cos\theta)\sin\phi\cos\frac{2\pi}{\lambda}(Vt-r)\text{ ....(1)},$$

where r is the distance of O from dS, θ and ϕ are the angles which r makes with the normal to dS drawn outwards and with the direction of vibration respectively, and $c\sin 2\pi Vt/\lambda$ is the displacement of the primary wave at the plane of resolution.

In all cases of diffraction, the illumination is insensible unless the inclination of r to the screen is small, which requires that θ should be small and ϕ nearly equal to $\frac{1}{2}\pi$; we may therefore as a sufficient approximation put $\cos\theta = \sin\phi = 1$, and the formula becomes

$$\frac{cdS}{\lambda r}\cos\frac{2\pi}{\lambda}(Vt-r)\text{ ....(2)},$$

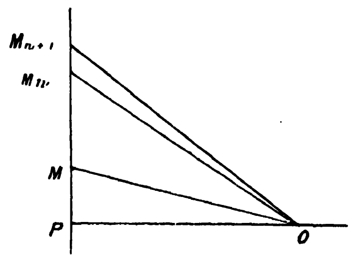
and the resultant vibration at O will be obtained by integrating this expression over the area of the aperture.

The formula (1), which is due to Stokes, will be hereafter rigorously deduced by means of a mathematical investigation, which is based on the assumption that the equations of motion of the luminiferous ether are of the same form as those of an elastic solid, whose power of resisting compression is very large in comparison with its power of resisting distortion. It will not

[1] Stokes, "On the Dynamical Theory of Diffraction;" *Trans. Camb. Phil. Soc.* vol. IX. p. 1; and *Math. and Phys. Papers*, vol. II. p. 243.

however be necessary at present to enter upon any investigations of this character, since all the leading phenomena of diffraction may be explained by means of the Principle of Huygens[1].

38. Let O be any point towards which a plane wave is advancing; draw $OP = r$ perpendicular to the front of the wave,



and with O as a centre describe a series of concentric spheres whose radii are $r + \frac{1}{2}\lambda, \ldots r + \frac{1}{2}n\lambda$. These spheres will divide the wave-front into a series of circular annuli, which are called Huygens' zones. Now $PM_n^2 = (r + \frac{1}{2}n\lambda)^2 - r^2$, and therefore, if λ^2 be neglected, $PM_n^2 = nr\lambda$, and the area of each zone is equal to $\pi r\lambda$.

Let $\cos\kappa(Vt - r)$, where $\kappa = 2\pi/\lambda$, be the displacement at O due to the original wave. Then it might be thought, that the displacement at O due to an element at M_n would be

$$A_n \cos\kappa(Vt - r - \tfrac{1}{2}n\lambda);$$

this however is not the case, inasmuch, as we shall presently see, that it is necessary to suppose the phases of successive elements to be different from that of the primary wave. Let e be this difference of phase; then since the amplitude of the zone is proportional to its area, the displacement produced by the nth zone will be

$$\pi r\lambda A_n \cos\kappa(Vt - r - \tfrac{1}{2}n\lambda + e) = \pi r\lambda\,(-)^n A_n \cos\kappa(Vt - r + e).$$

Accordingly the total displacement at O is

$$\pi r\lambda\,\{A_0 - A_1 + A_2 - \ldots + (-)^n A_n\} \cos\kappa(Vt - r + e).$$

Now the amplitude of the vibration produced at O by any

[1] The so-called Principle of Huygens is not a very satisfactory or rigorous method of dealing with the question of the resolution of waves. The reader may therefore, if he pleases, assume for the present the truth of Stokes' law. See Ch. XIII. and also, *Proc. Lond. Math. Soc.* vol. XXII. p. 317.

zone, is inversely proportional to its distance from O; we may therefore write $A_n = B_n/(r + \frac{1}{2}n\lambda)$, and the series becomes

$$\pi r\lambda \left\{\frac{B_0}{r} - \frac{B_1}{r+\frac{1}{2}\lambda} + \dots \frac{(-)^n B_n}{r+\frac{1}{2}n\lambda}\right\} \cos\kappa\,(Vt - r + e).$$

Now
$$\frac{B_{2n+1}}{r+(n+\frac{1}{2})\lambda} = \frac{B_{2n+1}}{r}\left\{1 - \frac{(n+\frac{1}{2})\lambda}{r}\right\};$$

also since B_{2n}, B_{2n+1}, B_{2n+2} are very nearly equal

$$\frac{B_{2n}}{r+n\lambda} + \frac{B_{2n+2}}{r+(n+1)\lambda} = \frac{2B_{2n+1}}{r}\left\{1 - \frac{(n+\frac{1}{2})\lambda}{r}\right.$$

$$= \frac{2B_{2n+1}}{r+(n+\frac{1}{2})\lambda};$$

whence every term of the series is approximately neutralized by half the sum of the terms which immediately precede and succeed it; accordingly the effect of the wave upon a distant point O is almost entirely confined to half that of the central portion PM, which remains over uncompensated.

It therefore follows that the displacement at O is equal to

$$\pi B_0 \int_r^{r+\frac{1}{2}\lambda} \cos\frac{2\pi}{\lambda}(Vt - r + e)\,dr = B_0\lambda \sin\frac{2\pi}{\lambda}(Vt - r + e).$$

Since this expression must be equal to the displacement produced at O by the primary wave, we must have $B_0\lambda = 1$, $e = \frac{1}{4}\lambda$. We thus obtain the important theorem that the displacement produced at O by any element dS of the primary wave

$$\cos\frac{2\pi}{\lambda}(Vt - r)$$

is equal to
$$-\frac{dS}{\lambda r}\sin\frac{2\pi}{\lambda}(Vt - r) \quad\text{.....................}\quad (3).$$

This result may also be obtained, as was done by Archibald Smith[1], by integrating over the whole wave-front, for

$$\iint \sin\frac{2\pi}{\lambda}(Vt - r)\frac{dS}{\lambda r} = \frac{2\pi}{\lambda}\int_r^\infty \sin\frac{2\pi}{\lambda}(Vt - r)\,dr$$

$$= -\cos\frac{2\pi}{\lambda}(Vt - r),$$

provided we suppose that $\cos\infty = 0$, an assumption which is justified by the result.

[1] *Camb. Math. Journ.* vol. III. p. 46.

39. We must now investigate the corresponding result in the case of a cylindrical wave.

If in (3) we write $R=(r^2+z^2)^{\frac{1}{2}}$ for r, and integrate with respect to z between the limits ∞ and $-\infty$, we shall obtain the effect produced by an infinite linear source at a point O, whose distance from the source is r; whence the effect of the source is equal to the real part of

$$\frac{\iota\delta y}{\lambda}\int_{-\infty}^{\infty}\epsilon^{-\iota\kappa(R-Vt}\frac{dz}{R}=\frac{2\iota\delta y}{\lambda}\epsilon^{\iota\kappa Vt}\int_{r}^{\infty}\frac{\epsilon^{-\iota\kappa R}dR}{(R^2-r^2)^{\frac{1}{2}}}.$$

If $u=R-r$, the integral becomes

$$\int_0^{\infty}\frac{\epsilon^{-\iota\kappa r-\iota\kappa u}}{u^{\frac{1}{2}}(2r+u)^{\frac{1}{2}}}du$$

$$=\frac{\epsilon^{-\iota\kappa r}}{(2r)^{\frac{1}{2}}}\int_0\frac{\epsilon^{-\iota\kappa u}}{\sqrt{u}}\left\{1-\frac{u}{2\,.\,2r}+\dots\frac{(-1)^n\,.\,1\,.\,3\dots(2n-1)}{2^n\,.\,n\,!}\left(\frac{u}{2r}\right)^n+\right\}du.$$

Also
$$\int_0^{\infty}\epsilon^{-\iota\kappa u}u^{n-\frac{1}{2}}\,du=2\int_0^{\infty}\epsilon^{-\iota\kappa x^2}x^{2n}dx$$

$$=\frac{1\,.\,3\,\dots\,(2n-1)\sqrt{\pi}}{2^n\,(\iota\kappa)^{n+\frac{1}{2}}}.$$

Hence if κr is large, which is always the case at a great distance from the source, all the terms of the series after the first may be neglected, and the integral will be equal to

$$\iota\kappa) \quad \backslash 2\kappa)\,(1-\iota);$$

accordingly the effect of the source at a distance r, is the real part of

$$\frac{\iota\delta y}{(2r\lambda)^{\frac{1}{2}}}(1-\iota)\epsilon^{-\iota\kappa(r-Vt)}=\frac{\delta y}{(\lambda r)^{\frac{1}{2}}}\sin\kappa\left(r-Vt+\tfrac{1}{8}\lambda\right)\dots(4).$$

We can now prove by integration, that this expression reproduces the original wave. Writing x^2+y^2 for r^2, we have $ydy=rdr$,

$$\int_{-\infty}^{\infty}\frac{\epsilon^{-\iota\kappa r}}{\sqrt{r}}dy=2\int_x^{\infty}\frac{\epsilon^{-\iota\kappa r}r^{\frac{1}{2}}dr}{(r^2-x^2)^{\frac{1}{2}}}.$$

Putting $r-x=u$, this becomes

$$2\int_0^{\infty}\frac{\epsilon^{-\iota\kappa x-\iota\kappa u}(u+x)^{\frac{1}{2}}du}{\sqrt{u}\,(2x+u)^{\frac{1}{2}}}$$

whence expanding in powers of u/x, and supposing $\kappa\lambda$ to be large, the integral is approximately equal to

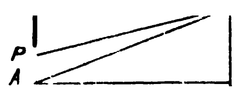
$$\frac{2\sqrt{\pi}\epsilon^{-\iota\kappa x}}{(1+\iota)\sqrt{\kappa}}.$$

Accordingly the total effect, which is the real part of

$$\frac{2\iota(1-\iota)\sqrt{\pi}}{(2\lambda)^{\frac{1}{2}}(1+\iota)\sqrt{\kappa}}\epsilon^{-\iota\kappa(x-Vt)} = \cos\kappa(x-Vt) \;\dots\dots\dots\; (5).$$

Diffraction through a Slit.

40. Before we discuss the general problem of diffraction, we shall consider the case in which plane waves are diffracted by a narrow slit in a screen, which is parallel to the wave-fronts.



Let A be the middle point of the slit, P any point on it, O any point on a screen on which the phenomena are observed. Let

$$AO = r, \quad PO = R, \quad AP = x, \quad PAO = \tfrac{1}{2}\pi - \theta.$$

If the vibration at O due to the element at A be

$$A\cos 2\pi\lambda^{-1}(r - Vt),$$

the vibration due to an element dx at P will be

$$A'\cos\frac{2\pi}{\lambda}(R - Vt)\,dx,$$

and therefore if a be the breadth of the aperture, the total disturbance at O will be

$$\zeta = \int_{-\frac{1}{2}a}^{\frac{1}{2}a} A'\cos\frac{2\pi}{\lambda}(R - Vt)\,dx.$$

Now $$R^2 = r^2 + x^2 - 2rx\sin\theta,$$

and since x is small compared with r, we have to a sufficient approximation

$$R = r - x\sin\theta.$$

Also the difference between A' and A may be neglected, whence

$$\zeta = A\int_{-\frac{1}{2}a}^{\frac{1}{2}a}\cos\frac{2\pi}{\lambda}(r - x\sin\theta - Vt)\,dx$$

$$= \frac{A\lambda}{\pi\sin\theta}\cos\frac{2\pi}{\lambda}(r - Vt)\sin\frac{\pi a\sin\theta}{\lambda},$$

and therefore the intensity is proportional to

$$I^2 = \frac{A^2\lambda^2}{\pi^2\sin^2\theta}\sin^2\frac{\pi a\sin\theta}{\lambda} \quad\text{..................(6).}$$

When $\theta = 0$, $I^2 = A^2a^2$, and consequently the projection of the central line of the slit is bright. The intensity is zero when $\sin\theta = m\lambda/a$, where m is any integer, and consequently the central bright band is surrounded by a series of dark bands. Putting $\pi a\lambda^{-1}\sin\theta = u$, it follows that the intensity is proportional to $u^{-2}\sin^2 u$, and the positions of the bright bands will be found by obtaining the maxima values of this expression. Equating the value of dI^2/du to zero, we shall obtain

$$\frac{\sin u}{u}\cdot\frac{u\cos u - \sin u}{u^2} = 0.$$

The first factor corresponds to the minima, and gives $u = m\pi$, the second gives

$$\tan u = u \quad\text{..........................(7).}$$

The roots of (7), as Lord Rayleigh has shown[1], may be calculated in the following manner.

Assume $$u = (m + \tfrac{1}{2})\pi - y = U - y,$$

where y is a positive quantity, which is small when u is large. Substituting in (7), we obtain

$$\cot y = U - y,$$

whence $$\tan y = \frac{1}{U}\left(1 + \frac{y}{U} + \frac{y^2}{U^2} + \ldots\ldots\right).$$

Expanding $\tan y$ in powers of y, we obtain

$$y = \frac{1}{U}\left(1 + \frac{y}{U} + \frac{y^2}{U^2} + \ldots\right) - \frac{y^3}{3} - \frac{2y^5}{15} - \frac{17y^7}{315}\ldots$$

[1] *Theory of Sound*, vol. I. § 207.

This equation is to be solved by successive approximation, from which it will be found that

$$u = U - y = U - U^{-1} - \frac{2}{3} U^{-3} - \frac{13}{15} U^{-5} - \frac{146}{105} U^{-7} \quad \ldots(8).$$

The values of u/π will thus be found to be 1·4303, 2·4590, 3·4709, 4·4747, 5·4818, 6·4844, &c. They were first obtained by a different method by Schwerd[1].

Since the maxima occur when u is nearly equal to $(m+\frac{1}{2})\pi$, it follows that the ratio of the intensities of successive bands to the central band is approximately equal to

$$\frac{4}{9\pi^2}, \quad \frac{4}{25\pi^2}, \quad \frac{4}{49\pi^2} \quad \&c.$$

We therefore see that the image formed by a slit does not consist of a bright band bounded by the edges of the geometrical shadow, but of a central bright band, surrounded by a number of alternately dark and bright bands.

Since the minima are determined by the equation

$$\sin\theta = m\lambda/a,$$

where θ is very small, it follows that the angular distance between two dark bands is λ/a. The bands are therefore broadest for red light and narrowest for violet light. Hence when sunlight is employed, a series of brilliantly coloured bands will be observed, which will however be necessarily limited in number, owing to the overlapping of the spectra of different colours.

41. It has already been stated, that one of the objections brought against the undulatory theory in its infancy was, that inasmuch as sound is known to be produced by aerial waves, it ought to follow that light should be able to bend round corners as sound is known to do, and that an obstacle ought not to be able to produce a distinct shadow. The results of the last article furnish an explanation of this apparent difficulty; for if λ be large compared with a, as is usually the case with sound, (6) becomes

$$I^2 = A^2a^2,$$

which is independent of θ, and consequently the intensity will be approximately constant for a considerable distance beyond the limits of the geometrical shadow. If on the other hand a is

[1] *Die Beugungserscheinungen*, Mannheim, 1835.

comparable with λ, the intensity will be insensible unless θ is small, and diffraction will take place.

The more general problem of diffraction through a large number of slits, will be discussed under the Theory of Gratings.

42. We shall now proceed to consider the general problem when the form of the aperture is given.

When light proceeding from a source, passes through a small aperture of any form, there are three possible cases to consider; according as the waves are (i) converging towards a focus in front of the aperture, (ii) are plane, (iii) are diverging from a focus behind the aperture; and as the analytical treatment of these three cases is different, we shall consider the first two cases in the present chapter, reserving the discussion of the third case for the succeeding one.

In the first case, let the screen upon which the phenomena are observed, pass through O, the focus towards which the light is converging; and let it be parallel to the plane of the aperture.

Let ξ, η be the coordinates of any point P on the screen referred to rectangular axes through O, and let x, y, z be any point Q of the aperture; also let f be the radius of the spherical wave at the aperture.

Then
$$\begin{aligned} QP^2 &= (x-\xi)^2 + (y-\eta)^2 + z^2 \\ &= f^2 - 2x\xi - 2y\eta + \xi^2 + \eta^2. \end{aligned}$$

Since ξ, η are very small compared with f, we may omit ξ^2, η^2, whence

$$QP = f - \frac{x\xi + y\eta}{f}.$$

Now if $\cos \kappa Vt$ be the vibration at the aperture, the vibration produced at P by an element dS at Q will be equal to

$$-\frac{dS}{\lambda f} \sin \kappa \left\{ Vt - f + \frac{x\xi + y\eta}{f} \right\}.$$

Integrating this over the area of the aperture, we shall find that the intensity at P is proportional to

$$= \frac{1}{\lambda^2 f^2} \left\{ \iint \sin \frac{2\pi}{\lambda f} (x\xi + y\eta)\, dx dy \right\}^2 + \frac{1}{\lambda^2 f^2} \left\{ \iint \cos \frac{2\pi}{\lambda f} (x\xi + y\eta)\, dx dy \right\}^2 \quad \text{...........(9)}.$$

Exactly the same result may be proved to hold good in the case of plane waves, provided f denotes the distance of the aperture from the screen; for in this case $x^2 + y^2$ may be neglected, and $z = f$.

We shall now discuss several cases.

Rectangular Aperture.

43. Let the aperture be a rectangle, whose sides are the lines $x = \pm \frac{1}{2}a$, $y = \pm \frac{1}{2}b$; then by integrating (9), we shall find that

$$I^2 = \frac{a^2b^2}{\lambda^2 f^2} \frac{\sin^2 \pi a\xi/\lambda f . \sin^2 \pi b\eta/\lambda f}{(\pi a\xi/\lambda f)^2 . (\pi b\eta/\lambda f)^2} \text{ .... } (10).$$

Each of these factors is of the form $u^{-1} \sin u$, whence the minima values of the intensity are given by $u = m\pi$, where m is any positive or negative integer, and accordingly the field is crossed by a series of dark lines whose equations are

$$\xi = m\lambda f/a, \quad \eta = m\lambda f/b \text{ .... } (11).$$

The intensity is evidently a maximum when $\xi = 0$, $\eta = 0$, in which case $I^2 = (ab/\lambda f)^2$.

To find the other maxima values, we observe that I^2 is the product of two factors of the form $u^{-2} \sin^2 u$; accordingly the maxima values are the roots of the equation

$$\tan u = u,$$

which has already been discussed in § 40.

The diffraction pattern accordingly consists of a central bright spot, surrounded by a series of dark lines whose equations are

$$\xi = m\lambda f/a, \quad \eta = m\lambda f/b;$$

and within the rectangle formed by consecutive dark lines, the intensity rises to a maximum; but these secondary maxima are far less bright than the centre of the pattern.

An Isosceles Triangle.

44. Let the vertex of the triangle be the origin, and let the axis of x be perpendicular to the base; also let e be the length of the perpendicular drawn from the vertex to the base, and let the

equations of the sides of the triangle be $y = \pm mx$. Then if $c = 2\pi e/\lambda f$,

$$\iint \sin \frac{2\pi}{\lambda f} (x\xi + y\eta)\, dx dy = \frac{\lambda^2 f^2}{4\pi^2 \eta} \left\{ \frac{\sin(\xi - m\eta) c}{\xi - m\eta} - \frac{\sin(\xi + m\eta) c}{\xi + m\eta} \right\},$$

and

$$\iint \cos \frac{2\pi}{\lambda f} (x\xi + y\eta)\, dx dy = \frac{\lambda^2 f^2}{4\pi^2 \eta} \left\{ \frac{1 - \cos(\xi + m\eta) c}{\xi + m\eta} - \frac{1 - \cos(\xi - m\eta) c}{\xi - m\eta} \right\},$$

accordingly the intensity

$$I^2 = \frac{\lambda^2 f^2}{4\pi^4 \eta^2} \left\{ \frac{\sin^2 \frac{1}{2} (\xi + m\eta) c}{(\xi + m\eta)^2} + \frac{\sin^2 \frac{1}{2} (\xi - m\eta) c}{(\xi - m\eta)^2} - \frac{\cos m\eta c (\cos m\eta c - \cos \xi c)}{\xi^2 - m^2 \eta^2} \right. \quad \text{(12)}.$$

Along the axis of y, $\xi = 0$, and the intensity becomes

$$I^2 = \frac{\lambda^2 f^2}{\pi^4 m^2 \eta^4} \sin^4 \tfrac{1}{2} m\eta c,$$

which at the origin is equal to

$$I^2 = m^2 e^4 / \lambda^2 f^2 \quad \text{(13)}.$$

The intensity at any point on the axis of x may be found either by evaluating (12) when $\eta = 0$, or directly from (9); we shall thus obtain

$$I^2 = \frac{m^2}{\pi^2 \xi^2} \left\{ e^2 + \frac{\lambda^2 f^2}{\pi^2 \xi^2} \sin^2 \frac{\pi \xi e}{\lambda f} - \frac{\lambda f e}{\pi \xi} \sin \frac{2\pi \xi e}{\lambda f} \right\}.$$

The case of an equilateral triangle may be worked out in a similar manner; but it will be more convenient to suppose that the origin is the centre of gravity of the triangle, so that the diffraction pattern is symmetrical with respect to the angular points of the triangle. The pattern exhibits a star-shaped appearance, which has been described by Sir J. Herschel[1].

Circular Aperture[2].

45. Let the aperture be a circle of radius c; also let

$$2\pi\xi/\lambda f = p, \quad 2\pi\eta/\lambda f = q.$$

Then by (9)

$$\lambda^2 f^2 I^2 = S^2 + C^2,$$

[1] *Encyclop. Metrop.* Art. Light, § 172.
[2] Airy, *Trans. Camb. Phil. Soc.* 1834.

where
$$S = \iint \sin(px + qy)\, dx dy,$$
$$C = \iint \cos(px + qy)\, dx dy,$$
the integration extending over the area of the circle.

To evaluate these integrals, we shall employ the theorem[1], that

$$\iint F(px + qy)\, dx dy = 2\int_{-c}^{c} (c^2 - x^2)^{\frac{1}{2}} F\{(p^2 + q^2)^{\frac{1}{2}} x\}\, dx, \text{ or } 0 \ldots (14),$$

according as F is an even or an odd function, where the double integral is taken over a circle of radius c.

Let r be the distance of any point of the screen from the projection of the centre of the aperture; then $2\pi r/\lambda f = (p^2 + q^2)^{\frac{1}{2}}$; whence by (14), $S = 0$ and

$$C = 2\int_{-c}^{c} (c^2 - x^2)^{\frac{1}{2}} \cos(2\pi r x/\lambda f)\, dx$$
$$= 2c^2 \int_0^{\pi} \sin^2\phi \cos(2\pi c r \cos\phi/\lambda f)\, d\phi$$
$$= 2\pi c^2 \frac{J_1(2\pi c r/\lambda f)}{2\pi c r/\lambda f},$$

whence
$$I^2 = \frac{4\pi^2 c^4}{\lambda^2 f^2}\left\{\frac{J_1(2\pi c r/\lambda f)}{2\pi c r/\lambda f}\right\}^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (15),$$

where J_1 is Bessel's function of order unity.

The properties of Bessel's functions have been discussed by the various writers referred to below[2]. The function $J_1(x)$ may be expressed in either of the forms

$$J_1(x) = \frac{x}{\pi}\int_0^{\pi} \cos(x\cos\phi)\sin^2\phi\, d\phi \ldots\ldots\ldots\ldots (16),$$

or
$$J_1(x) = \frac{x}{2} - \frac{x^3}{2^2 . 4} + \frac{x^5}{2^2 . 4^2 . 6} - \ldots\ldots\ldots\ldots (17).$$

It is also known that

$$\left.\begin{aligned} J_0' &= -J_1 \\ 2J_n' &= J_{n-1} - J_{n+1} \\ \frac{2nJ_n}{x} &= J_{n-1} + J_{n+1} \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (18).$$

From (15) and (17) we see that when $r = 0$,

$$I^2 = \pi^2 c^4/\lambda^2 f^2;$$

[1] The theorem may be proved by turning the axes through an angle $\tan^{-1} p/q$.

[2] Lommel, *Bessel'sche Functionen*. Lord Rayleigh, *Theory of Sound*. Todhunter, *Functions of Laplace, Lamé and Bessel*. Heine, *Kugelfunctionen*.

the origin is therefore a bright spot, whose intensity is proportional to the fourth power of the radius of the aperture.

Writing x for $2\pi cr/\lambda f$, it follows that the minima are determined by the equation $J_1(x)=0$. The roots of the equation $J_1(x/\pi)=0$ have been calculated by Stokes[1], and are equal to 1·2197, 2·2330, 3·2383, 4·2411, 5·2428, 6·2439, &c.; from which it appears that the first dark ring occurs when

$$r/f = 1{\cdot}2197 \times \lambda/2c.$$

Since λ is very small, it is necessary that the radius of the aperture should be small in order that this ring should be seen distinctly.

The maxima are determined by the equation

$$\frac{d}{dx}\left(\frac{J_1}{x}\right) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(19);$$

whence by the last two of (18), the maxima are determined by the roots of the equation $J_2(x)=0$. The value of the intensity in this case is

$$\frac{\pi^2 c^4}{\lambda^2 f^2} J_0^2(x).$$

The following table, which has been calculated by Lommel, gives the values of x for which $J_2(x)=0$, and the corresponding values of $J_0^2(x)$.

	$J_0^2(x)$
·00000	1·00000
5·13563	·01750
8·41724	·00416
11·61986	·00160
14·79594	·00078

From this table, it appears that the maximum intensity of the first bright ring is only about $\frac{1}{57}$th of that of the central spot. The diffraction pattern therefore consists of a central bright spot, surrounded by a series of dark and bright rings; moreover the central spot is the brightest, and by far the greater portion of the whole illumination is concentrated in it.

[1] *Trans. Camb. Phil. Soc.* vol. IX. p. 166.

Elliptic Aperture.

46. The corresponding results for an elliptic aperture can be obtained in a similar manner.

Let the equation of the ellipse be $x^2/a^2 + y^2/b^2 = 1$, and let $x' = x/a$, $y' = y/b$, $p' = pa$, $q' = qb$. Then the values of S and C will be

$$S = ab \iint \sin (p'x' + q'y')\, dx'dy'$$
$$C = ab \iint \cos (p'x' + q'y')\, dx'dy',$$

where the integration extends over the circle $x'^2 + y'^2 = 1$. Whence $S = 0$ and

$$C = 2ab \int_{-1}^{1} (1 - x^2)^{\frac{1}{2}} \cos (p'^2 + q'^2)^{\frac{1}{2}} x dx$$
$$= \frac{2}{u} \pi ab J_1 (u),$$

where $$u^2 = p'^2 + q'^2 = \frac{4\pi^2}{\lambda^2 f^2} (a^2\xi^2 + b^2\eta^2).$$

It therefore follows that the curves of constant intensity are similar to the reciprocal ellipse $a^2x^2 + b^2y^2 = a^2b^2$.

Talbot's Bands.

47. These bands were first observed by Fox Talbot[1], and are produced when a tolerably pure spectrum is viewed by a telescope, *half the aperture of which is covered by a thin plate of glass or mica.*

The theory of these bands was first given by Airy[2], but we shall follow the investigation of Stokes[3].

We shall suppose, that the object-glass of a telescope is limited by a screen, in which there is a rectangular aperture, the lengths of whose sides are $2l$ and $h + 2g + k$. Let k be the width of the thin plate, h that of the unretarded stream; we shall also suppose

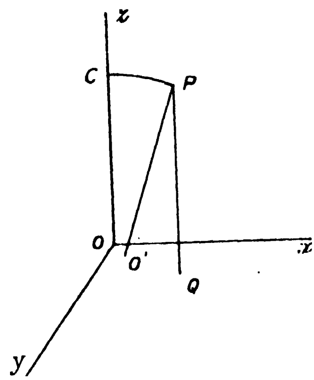
[1] *Phil. Mag.* vol. x. p. 364, 1837. Brewster, *Rep. of 7th Meeting of Brit. Assoc.*

[2] "On the Theoretical Explanation of an apparent new Polarity of Light," *Phil. Trans.* 1840 and 1841.

[3] "On the Theory of certain Bands seen in the Spectrum," *Phil. Trans.* 1848, p. 227; *Math. and Phy. Papers*, vol. II. p. 14.

that there is an opaque interval of width $2g$ between the two streams, and that the axis of the telescope passes through the centre of the opaque interval.

Let C be the centre of the opaque plate, O the projection of C on the focal plane of the object-glass; let O be the origin, and let the axes of x and y be respectively parallel to the sides $h+2g+k$ and $2l$ of the aperture.



We shall first consider the light which emanates from any point of a spectrum whose plane is parallel to the plane xy.

After passing through the object-glass of the telescope, the light emanating from this point will consist of a spherical wave, whose radius is equal to OC, which converges to a point O' as focus. Let P be any point on this wave, Q any point on the plane xy; let (x, y, z) be the coordinates of P; (p, q) those of O'; (ξ, η) those of Q; also let $OC=f$.

The displacement at Q due to an element at P is equal to

$$\frac{cdS}{\lambda PQ}\sin\frac{2\pi}{\lambda}(Vt-PQ).$$

Now
$$PQ^2=(\xi-x)^2+(\eta-y)^2+f^2-(x-p)^2-(y-q)^2$$
$$=f^2-2x(\xi-p)-2y(\eta-q),$$

if $\xi^2+\eta^2-p^2-q^2$ be neglected. Whence if

$$\xi'=\xi-p,\quad \eta'=\eta-q,$$

the displacement at Q becomes approximately

$$\frac{cdS}{\lambda f}\sin\frac{2\pi}{\lambda}(Vt-f+x\xi'/f+y\eta'/f)\ldots\ldots\ldots\ldots(20).$$

Since the thin plate of glass or mica occupies the space bounded by the lines $x=g+k$, $x=g$, $y=l$, $y=-l$; it follows that in order to obtain the resultant displacement at Q due to the whole wave produced by the point, we must change dS into $dxdy$, f into $f+R$, where R the retardation due to the plate is a small quantity whose square may be neglected, and integrate over the area of the thin plate, that is from $y=l$ to $-l$, and $x=g+k$ to g; whence performing the integration, we shall obtain

$$v=\frac{2cl}{\pi\xi'}\cdot\frac{\lambda f}{2\pi l\eta'}\sin\frac{2\pi l\eta'}{\lambda f}\sin\frac{\pi k\xi'}{\lambda f}\sin\frac{2\pi}{\lambda}\left(Vt-R-f+\frac{\xi' g}{f}+\frac{\xi' k}{2f}\right.\quad\text{.........}(21).$$

To obtain the displacement at Q due to the unretarded stream, we must put $R=0$, and integrate (20) from $y=l$ to $-l$, and $x=-g$ to $-g-h$; accordingly we obtain

$$v'=\frac{2cl}{\pi\xi'}\cdot\frac{\lambda f}{2\pi l\eta'}\sin\frac{2\pi l\eta'}{\lambda f}\sin\frac{\pi h\xi'}{\lambda f}\sin\frac{2\pi}{\lambda}\left(Vt-f-\frac{\xi' g}{f}-\frac{\xi' h}{2f}\right)\text{...}(22).$$

The total displacement at Q due to the two streams of light is equal to $v+v'$; accordingly if we put

$$Q=\frac{\lambda f}{2\pi l\eta'}\sin\frac{2\pi l\eta'}{\lambda f},\ \rho=2\pi R/\lambda$$

$$P=\sin^2\frac{\pi h\xi'}{\lambda f}+\sin^2\frac{\pi k\xi'}{\lambda f}+2\sin\frac{\pi h\xi'}{\lambda f}\sin\frac{\pi k\xi'}{\lambda f}\cos\left\{\rho-\frac{\pi\xi'}{\lambda f}(4g+h+k)\right.\quad\text{.........}(23),$$

the intensity I^2 will be given by the equation

$$I^2=\frac{4c^2l^2}{\pi^2\xi'^2}Q^2P\text{.......................}(24).$$

This is the expression for the intensity due to a point of light whose geometrical focus is O'.

48. To obtain the intensity of a line of homogeneous light which is parallel to the axis of y, we must write $Af^{-1}dq$ for c^2, and integrate from a large positive to a large negative value of q, the largeness being estimated in comparison with $\lambda f/l$. Now the angle $2\pi\eta' l/\lambda f$ changes by π when q changes by $\lambda f/2l$, which is therefore the breadth, in the direction of y, of one of the diffraction bands which would be seen with a luminous point. Since l is not supposed to be extremely small, but on the contrary moderately large, the whole system of diffraction bands would occupy but a very small portion of the field of view in the direction of y, so that

we may without sensible error suppose the limits of q to be $-\infty$ and ∞. Since P does not contain q, it follows from (24) that the resultant intensity due to a luminous line is

$$I^2 = \frac{4Al^2}{\pi^2\xi'^2 f} P \int_{-\infty}^{\infty} \left\{ \frac{\lambda f}{2\pi l(\eta - q)} \cdot \sin \frac{2\pi l(\eta - q)}{\lambda f} \right\}^2 dq$$

$$= \frac{2Al\lambda}{\pi^3\xi'^2} P \int_{-\infty}^{\infty} \frac{\sin^2\theta}{\theta^2} d\theta$$

$$= \frac{2Al\lambda}{\pi^2\xi'^2} P \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (25).$$

49. To find the total intensity at Q due to a plane area of homogeneous light, it would be necessary to change A into $Bf^{-1}dp$, and integrate with respect to p between the limits ∞ and $-\infty$; but since the bands which we are investigating are produced by a *spectrum*, the colour, and therefore B and ρ, vary from point to point. The variations of B and λ may however be neglected in the integration, except in the term ρ or $2\pi R/\lambda$, because a small variation of λ produces a comparatively large change of phase.

Since ρ depends upon the position of O', we shall have $\rho = f(p)$; whence if ρ' and ϖ denote the values of ρ and $-d\rho/d\xi$ at Q, we shall have

$$\rho = f\{\xi - (\xi - p)\}$$
$$= \rho' + \varpi(\xi - p) = \rho' + \varpi\xi'$$

approximately.

Let
$$\left.\begin{array}{c} \pi h/\lambda f = h', \ \pi k/\lambda f = k' \\ \varpi - \pi(4g + h + k)/\lambda f = g' \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (26);$$

also let $u = p - \xi = -\xi'$. Since $du = dp$, it follows from (23) and (25) that the intensity is determined by the equation

$$I^2 = \frac{2Bl\lambda}{\pi^2 f} \int_{-\infty}^{\infty} \{\sin^2 h'u + \sin^2 k'u + 2\sin h'u \sin k'u \cos(\rho' - g'u)\} \frac{du}{u^2}$$
$$\ldots\ldots\ldots\ldots (27).$$

Now
$$\int_{-\infty}^{\infty} u^{-2} \sin^2 h'u\, du = \pi h';$$

also if we write

$$\cos(\rho' - g'u) = \cos\rho' \cos g'u + \sin\rho' \sin g'u,$$

the portion of the integral in (27) which involves $\sin\rho'$ will vanish,

because the positive and negative elements destroy one another; accordingly

$$I^2 = \frac{2Bl\lambda}{\pi f}\left(h' + k' + \frac{2w}{\pi}\cos\rho'\right) \text{..............(28),}$$

where $$w = \int_{-\infty}^{\infty} u^{-2} \sin h'u \sin k'u \cos g'u du.$$

Differentiating with respect to g', we get

$$\frac{dw}{dg'} = -\int_{-\infty}^{\infty} u^{-1} \sin h'u \sin k'u \sin g'u du$$

$$= \tfrac{1}{4}\int_{-\infty}^{\infty} \{\sin(g' + h' + k')u + \sin(g' - h' - k')u$$
$$- \sin(g' + h' - k')u - \sin(g' + k' - h')u\} u^{-1} du.$$

But it is well known that

$$\int_{-}^{\circ} u^{-1} \sin su du = \pi \text{ or } -\pi,$$

according as s is positive or negative. If therefore we use $F(s)$ to denote the discontinuous function which is represented by the above integral, and which is equal to π or $-\pi$ according as s is positive or negative, we get

$$dw/dg' = \tfrac{1}{4}\pi\,\{F(g' + h' + k') + F(g' - h' - k') - F(g' + h' - k')$$
$$- F(g' + k' - h')\}.$$

This equation gives

$dw/dg' = 0$, from $g' = -\infty$ to $g' = -(h' + k')$;
$= \frac{1}{2}\pi$, from $g' = -(h' + k')$ to $g' = -(h' \sim k')$;
$= 0$, from $g' = -(h' \sim k')$ to $g' = +(h' \sim k')$;
$= -\frac{1}{2}\pi$, from $g' = h' \sim k'$ to $g' = h' + k'$;
$= 0$, from $g' = h' + k'$ to $g = \infty$:

the sign $\sim$ being used to denote the difference between h' and k' when $h' > k'$; if $h' < k'$, the expression $h' \sim k'$ denotes $k' - h'$.

Now w vanishes when $g' = \pm\infty$, on account of the fluctuations of the factor $\cos g'u$ under the integral sign; whence integrating the value of dw/dg' given above, and determining the constant of integration, so that $w = 0$ when $g' = -\infty$, we obtain

$w = 0$, from $g' = -\infty$ to $g' = -(h' + k')$;
$w = \frac{1}{2}\pi(h' + k' + g')$, from $g' = -(h' + k')$ to $g' = -(h' \sim k')$;
$w = \pi k'$ or $\pi h'$ (according as $h' > k'$ or $h' < k'$)
from $g' = -(h' \sim k')$ to $g' = +(h' \sim k')$;
$w = \frac{1}{2}\pi(h' + k' - g')$, from $g' = h' \sim k'$ to $g' = h' + k'$;
$w = 0$, from $g' = h' + k'$ to $g' = \infty$.

Substituting, in (28), and putting $g' = \pi\mathfrak{g}/\lambda f$ in the last of (26), so that

$$\mathfrak{g} = \varpi\lambda f/\pi - 4g - h - k \quad\text{..................(29)},$$

we get the following three expressions for the intensity

(i) When the numerical value of $\mathfrak{g}$ exceeds $h+k$

$$I^2 = 2Blf^{-2}(h+k) \quad\text{.......................(30).}$$

(ii) When the numerical value of $\mathfrak{g}$ lies between $h+k$ and $h-k$

$$I^2 = 2Blf^{-2}\{h+k+(h+k-\sqrt{\mathfrak{g}^2})\cos\rho'\} \quad\text{.........(31).}$$

(iii) When the numerical value of $\mathfrak{g}$ is less than $h \sim k$,

$$I^2 = 2Blf^{-2}(h+k+2h\cos\rho') \text{ or } 2Blf^{-2}(h+k+2k\cos\rho') \quad\text{...(32)}$$

according as h or k is the smaller of the two.

50. In discussing these results, Sir G. Stokes says:

"Let the axis of x be always reckoned positive in the direction in which the blue end of the spectrum is seen, so that in the image formed at the focus of the object-glass or on the retina, according as the retarding plate is placed in front of the object-glass or in front of the eye, the blue is on the negative side of the red. Although the plate has been supposed at the positive side, there will be no loss of generality, for should the plate be at the negative side it will only be requisite to change the sign of ρ.

"First, suppose ρ to decrease algebraically in passing from the red to the blue. This will be the case when the retarding plate is held at the side on which the red is seen. In this case ϖ is negative, and therefore $\mathfrak{g} < -(h+k)$, and therefore (30) is the expression for the intensity. This expression indicates a uniform intensity, so that there are no bands at all.

"Secondly, suppose ρ to increase algebraically in passing from the red to the blue. This will be the case when the retarding plate is held at the side on which the blue is seen. In this case ϖ is positive; and since ϖ varies as the thickness of the plate, $\mathfrak{g}$ may be made to assume any value from $-(4g+h+k)$ to $+\infty$ by altering the thickness of the plate.

51. "The plate being placed as in the preceding paragraph, suppose first that the breadths h, k of the interfering streams are equal and that the streams are contiguous, so that $g=0$. Then the expression (32) may be dispensed with, since it only holds

good when $\mathfrak{g}=0$, in which case it agrees with (31). Let T_0 be the value of the thickness T for which $\mathfrak{g}=0$. Then $T=0$ corresponds to $\mathfrak{g}=-(h+k)$, $T=T_0$ to $\mathfrak{g}=0$, and $T=2T_0$ to $\mathfrak{g}=h+k$; and for values of T equidistant from T_0, the values of $\mathfrak{g}$ are equal in magnitude but of opposite signs. Hence, provided T be less than $2T_0$, there are dark and bright bands formed, the vividness of the bands being so much the greater as T is more nearly equal to T_0, for which particular value the minima are absolutely black.

"Secondly, suppose the breadths h, k of the two streams to be equal as before, but suppose the streams separated by an interval $2g$; then the only difference is that $\mathfrak{g}=-(h+k)$ corresponds to a positive value, T_2 suppose, of T. If T be less then T_2, or greater than $2T_0-T_2$, there are no bands; but if T lie between T_2 and $2T_0-T_2$ bands are formed, which are most vivid when $T=T_0$, in which case the minima are perfectly black.

"Thirdly, suppose the breadths h, k of the interfering streams unequal, and suppose, as before, that the streams are separated by an interval $2g$; then $\mathfrak{g}=-(h+k)$ corresponds to a positive value, T_2 suppose, of T: $\mathfrak{g}=-(h\sim k)$ corresponds to another positive value, T_1 suppose, of T, T_1 lying between T_2 and T_0, T_0 being as before the value of T which gives $\mathfrak{g}=0$. As T increases from T_0, $\mathfrak{g}$ becomes positive and increases from 0, and becomes equal to $h\sim k$ when $T=2T_0-T_1$, and to $h+k$ when $T=2T_0-T_2$. When $T<T_2$ there are no bands. As T increases to T_1 bands become visible, and increase in vividness till $T=T_1$, when the ratio of the minimum intensity to the maximum becomes that of $h-k$ to $h+3k$, or of $k-h$ to $k+3h$, according as $h >$ or $< k$. As T increases to $2T_0-T_1$, the vividness of the bands remains unchanged; and as T increases from $2T_0-T_1$ to $2T_0-T_2$, the vividness decreases by the same steps as it increased. When $T=2T_0-T_2$, the bands cease to exist, and no bands are formed for a greater value of T.

"The particular thickness T_0 may be conveniently called the *best thickness*. This term is to a certain extent conventional, since when h and k are unequal the thickness may range from T_1 to $2T_0-T_1$ without any change being produced in the vividness of the bands. The best thickness is determined by the equation

$$\varpi=-\frac{d\rho}{dp}=\frac{\pi}{\lambda f}(4g+h+k).$$

Now in passing from one band to its consecutive, ρ changes by 2π, and p by e, if e be the linear breadth of a band; and for this small change of p we may suppose the changes in ρ and ξ proportional, or put $-d\rho/dp = 2\pi/e$. Hence the best aperture for a given thickness is that for which

$$4g + h + k = 2\lambda f/e.$$

If $g = 0$, and $k = h$, this equation becomes $h = \lambda f/e$."

The theory of Talbot's bands with a half covered *circular* aperture has been discussed by H. Struve[1].

Resolving Power of Optical Instruments.

52. When a distant object is viewed through a telescope, an image of the object is formed at the focus of the object-glass which is magnified by the eye-glass; and in order that the object should appear well defined, it is necessary that each point of it should form a sharp image. The indefiniteness which is sometimes observed in images is partly due to aberration; this however can in great measure be got rid of by proper optical appliances, but there is another cause, viz. diffraction, which also produces indefiniteness, as we shall proceed to show.

If we suppose that the aperture of the telescope is a rectangle, it appears from § 43, that the intensity at the focal point is equal to $(ab/\lambda f)^2$, and therefore increases as the dimensions of the aperture increase; on the other hand the distances between the dark lines parallel to x and y are respectively equal to $\lambda f/a$ and $\lambda f/b$, and therefore diminish as a and b increase; accordingly the diffraction pattern becomes almost invisible as the aperture increases, and the bright central spot alone remains. The effect of a large aperture is consequently to diminish the effect of diffraction, and to increase the definition of an image.

When two very distant objects, such as a double star, are viewed by the naked eye, the two objects are undistinguishable from one another, and only one object appears to be visible. If however the two objects are viewed through a telescope, it frequently happens that both objects are seen, owing to the fact that the telescope is able to separate or resolve them; and it

[1] *St Petersburg Trans.* vol. XXXI. No. 1, 1883.

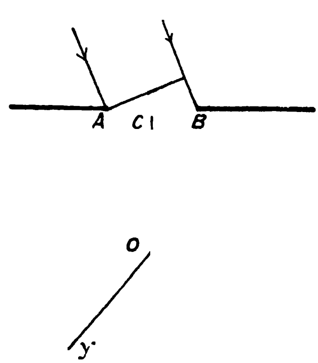
might at first sight appear, that a telescope of sufficient power would be capable of resolving two objects however distant they might be. This however is not the case, owing to the fact that the finiteness of the wave-length of light, coupled with the impossibility of constructing telescopes of indefinitely large dimensions, impose a limit to the resolving powers of the latter.

According to geometrical optics, an image of each double star will be formed at two points which very nearly coincide with the principal focus of the object-glass; but physical optics shows that two diffraction patterns will be formed, whose centres are the geometrical images of each star. If the two diffraction patterns overlap to such an extent, that the appearance consists of a patch of light of variable intensity in which the two central bright spots are undistinguishable, the double star will not be resolved; but if the two patterns do not overlap to such an extent as to make the central spots undistinguishable, the double star will be resolved into its two components.

53. In order to investigate this question mathematically, and at the same time to simplify the analysis as much as possible, we shall suppose that the light from each star consists of plane waves which make an angle θ with the plane xy; and we shall investigate the intensity at points on the axis of x. If

$$-(\lambda r)^{-1} \sin \kappa\,(Vt - r)\, dS$$

denote the displacement produced at P, by the element of the



wave which is situated at the centre C of the aperture, the displacement produced by any other element will be

$$-\frac{dxdy}{\lambda r} \sin \frac{2\pi}{\lambda} \{Vt - R - x \sin \theta\}.$$

Now if $OC = f$,

$$R = f - x\xi/f,$$

whence the total displacement at P is

$$-\frac{b}{\lambda f}\int_{-\frac{1}{2}a}^{\frac{1}{2}a} \sin\frac{2\pi}{\lambda}\{Vt - f + x(\xi - f\sin\theta)/f\}\,dx$$

and consequently the intensity will be proportional to

$$\frac{b^2}{\pi^2(\xi - f\sin\theta)^2}\sin^2\frac{\pi a}{\lambda f}(\xi - f\sin\theta) \quad\text{.........}\quad (33).$$

The greatest maximum value of this expression occurs when

$$\xi = f\sin\theta \quad\text{..........................}\quad (34),$$

which gives the position of the central bright spot; and the first minimum, which occurs on the negative side of this point, is given by

$$\xi' = f(\sin\theta - \lambda/a) \quad\text{....................}\quad (35).$$

The intensity due to the other component of the double star will be obtained by changing ξ into $-\xi$; accordingly, the greatest maximum will occur when $\xi = \xi''$, where

$$\xi'' = -f\sin\theta \quad\text{.......................}\quad (36).$$

Let us now suppose that the first minimum of the diffraction pattern due to the left-hand component of the double star, coincides with the greatest maximum of the right-hand component; then $\xi' = \xi''$, whence by (35) and (36)

$$2\sin\theta = \lambda/a \quad\text{.........................}\quad (37),$$

or since θ is very small,

$$2\theta = \lambda/a \quad\text{..........................}\quad (38).$$

By (33), the value of the intensity at either of the bright points is a^2b^2/λ^2f^2; and the intensity of either component at $\xi = 0$ is

$$\frac{b^2}{\pi^2 f\sin^2\theta}\sin^2\left(\frac{\pi a}{\lambda}\sin\theta\right) = \frac{4a^2b^2}{\pi^2\lambda^2f^2}$$

by (37); whence the ratio of the intensity at the middle point to that of either of the bright points is equal to $8/\pi^2 = \cdot 8106$.

It thus appears that the brightness midway between the two geometrical images, is about $\frac{4}{5}$ths of the brightness of the images themselves; and from experiment it appears that this is about the limit at which there could be any decided appearance of resolution. Now 2θ is the angle which the components of the double star subtend at the place of observation; and since by (38) $2\theta = \lambda/a$,

we see that *an object cannot be resolved, unless its components subtend at the place of observation, an angle which exceeds that subtended by the wave-length of light, at a distance equal to the breadth of the aperture.*

If the distance of the object be such that $\theta = \lambda/a$, it appears from (35) that $\xi' = 0$; there is accordingly a dark band at the middle point of the two images, which is more than sufficient for resolution.

54. We shall now consider the resolving power of a telescope having a circular aperture.

Let the axis of ξ be drawn perpendicularly to the line of intersection of the fronts of the waves with the plane of the aperture. Then at points on the axis of ξ, the intensity due to that component of a double star, which lies on the left-hand side will be

$$I^2 = C^2 + S^2,$$

where
$$C = \frac{1}{\lambda f}\iint \cos\frac{2\pi x}{\lambda f}(\xi - f\sin\theta)\,dx\,dy,$$

$$S = \frac{1}{\lambda f}\iint \sin\frac{2\pi x}{\lambda f}(\xi - f\sin\theta)\,dx\,dy,$$

and the integration extends over the area of the aperture. From these expressions we see that $S = 0$, and

$$C = \frac{2\pi c}{\lambda f p} J_1(pc),$$

where $p = 2\pi(\xi - f\sin\theta)/\lambda f$, and c is the radius of the aperture.

The greatest maximum occurs when $p = 0$, or $\xi = f\sin\theta$, which gives the central spot; and the intensity at this point is equal to

$$\pi^2 c^4/\lambda^2 f^2.$$

The first dark band to the left of the central spot occurs when $J_1(pc) = 0$, or $pc/\pi = -1{\cdot}2197$; in which case

$$\xi' = f\sin\theta - (f\lambda/c)\times{\cdot}6098.$$

If ξ'' be the distance of the central spot due to the right-hand component,

$$\xi'' = -f\sin\theta;$$

accordingly if $\xi' = \xi''$,

$$2\sin\theta = (\lambda/c)\times{\cdot}6098 = \frac{\lambda}{\frac{5}{3}c}\text{ nearly}\ldots\ldots\ldots\ldots(39).$$

When $\xi = 0$,

$$C = \frac{c}{f\sin\theta} J_1\left(\frac{2\pi c}{\lambda}\sin\theta\right)\ldots\ldots\ldots\ldots\ldots\ldots(40),$$

and if θ is given by (39)

$$C = \frac{10c^2}{3f\lambda} J_1(1\cdot8849) = \frac{c^2}{f\lambda} \times 1\cdot937;$$

whence the ratio of the intensity at the middle point, to that of either bright spot is about $7\cdot5 \div \pi^2$. The corresponding number for a rectangular aperture was found to be $8/\pi^2$, and 2θ was equal to λ/a; whereas in the present case $2\theta = \lambda/\frac{5}{3}c$. If therefore the components of a double star subtend at the place of observation an angle, which is somewhat greater than the angle subtended by the wave-length of light, at a distance equal to the diameter $2c$ of the circular aperture, the telescope will resolve the star. Hence the resolving power of a telescope having a circular aperture, is less than one whose aperture is rectangular.

The resolution of a double line is discussed in Lord Rayleigh's article on Wave Theory[1].

Theory of Gratings[2].

55. A diffraction grating consists of a thin plate of glass, on which a very large number of fine lines have been ruled with a diamond very close together; and gratings have been constructed, which contain as many as 40000 lines to the inch. When light is incident upon the grating, the lines of the latter act the part of approximately opaque obstacles, and a diffraction spectrum is produced.

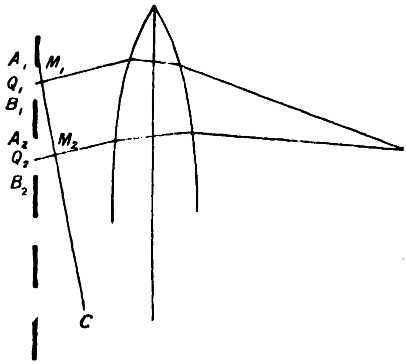
For the purpose of presenting the theory in its simplest form, we shall suppose that plane waves of light, whose fronts are parallel to the grating, fall upon the latter, and are then refracted by a convex lens, which is likewise parallel to the grating; and we shall examine the appearance on a screen which passes through the principal focus of the lens.

If we consider the system of parallel rays, which falls upon the lens and is perpendicular to any plane A_1C, which makes an angle θ with the grating, these rays will be brought to a focus by the lens at a point P, which is near the principal focus; and consequently each ray will occupy the same time in travelling from

[1] *Encyc. Brit.*

[2] Lord Rayleigh, Art. Wave Theory, § 14, *Encyc. Brit.* p. 437.

A_1C to P. Let Q_1, Q_2... be points on the transparent parts of the grating, whose distances from A_1, A_2... are equal to x; let $A_1B_1 = A_2B_2 = \dots a$; $B_1A_2 = B_2A_3 = \dots d$.



Then the resultant displacement at P is proportional to

$$\int_0^a \left[\sin 2\pi\left(\frac{t}{\tau}+\frac{x}{\lambda}\sin\theta\right)+\sin 2\pi\left(\frac{t}{\tau}+\frac{x+a+d}{\lambda}\sin\theta\right)\dots\right.$$
$$\left.+\sin 2\pi\left\{\frac{t}{\tau}+\frac{x+n(a+d)}{\lambda}\sin\theta\right\}\right]dx\dots(41),$$

where n is the number of opaque lines of the grating. Integrating, this becomes equal to

$$-\frac{\lambda}{2\pi\sin\theta}\left[\cos 2\pi\left(\frac{t}{\tau}+\frac{a}{\lambda}\sin\theta\right)-\cos\frac{2\pi t}{\tau}\right.$$
$$+\cos 2\pi\left(\frac{t}{\tau}+\frac{2a+d}{\lambda}\sin\theta\right)-\cos 2\pi\left(\frac{t}{\tau}+\frac{a+d}{\lambda}\sin\theta\right)$$
$$\left.+\cos 2\pi\left\{\frac{t}{\tau}+\frac{a+n(a+d)}{\lambda}\sin\theta\right\}-\cos 2\pi\left\{\frac{t}{\tau}+\frac{n(a+d)}{\lambda}\sin\theta\right\}\right]$$
$$\dots\dots(42).$$

If θ be chosen so that

$$(a+d)\sin\theta = m\lambda\dots\dots\dots\dots\dots\dots\dots(43),$$

where m is a positive integer, each of the $n+1$ pairs of terms become equal to one another, and the series is equal to

$$\frac{(n+1)\lambda}{2\pi\sin\theta}\left\{\cos\frac{2\pi t}{\tau}-\cos 2\pi\left(\frac{t}{\tau}+\frac{ma}{a+d}\right)\right\}\dots\dots(44).$$

It therefore appears, that for the directions which are determined by (43), the disturbances produced at P by the transparent

portions of the different elements of the grating reinforce one another, and that the intensity is a maximum for these directions. Accordingly when homogeneous light is employed, the diffraction spectrum consists of a number of bright and dark lines, and the bright lines occur when the position of A_1C is such that the projection of the element $a+d$ upon it is equal to any multiple of a wave-length. The central band is bright, and its intensity I_0^2 is equal to $(n+1)^2a^2$, as can at once be seen from (41) by putting $\theta=0$, and then performing the integration.

If I_m^2 be the intensity of the mth bright band, it follows from (43) and (44) that

$$I_m^2 = \frac{(n+1)^2(a+d)^2}{m^2\pi^2}\sin^2\frac{am\pi}{a+d},$$

whence

$$\frac{I_m^2}{I_0^2} = \left(\frac{a+d}{am\pi}\right)^2 \sin^2\frac{am\pi}{a+d} \qquad (45).$$

If the whole space occupied by the grating were transparent, the disturbance at P would be

$$\int_0^{a+n(a+d)} \sin 2\pi\left(\frac{t}{\tau}+\frac{x}{\lambda}\sin\theta\right)dx,$$

and the intensity would be proportional to

$$\frac{\lambda^2}{\pi^2\sin^2\theta}\sin^2\frac{\pi}{\lambda}\{a+n(a+d)\}\sin\theta;$$

accordingly the intensity in the direction $\theta=0$, is

$$I^2 = \{a+n(a+d)\}^2,$$

and therefore

$$\frac{I_m^2}{I^2} = \frac{(n+1)^2(a+d)^2}{m^2\pi^2\{a+n(a+d)\}^2}\sin^2\frac{am\pi}{a+d}$$

$$= \frac{1}{m^2\pi^2}\sin^2\frac{am\pi}{a+d} \qquad (46),$$

if the number of lines is so large that n may be treated as infinite.

Since the sine of an angle can never be greater than unity, it follows that the amount of light in the mth spectrum can never be greater than $1/m^2\pi^2$ of the original light. Hence in a grating composed of alternately opaque and transparent parts, whose breadths are equal, so that $d=a$, the central image is the brightest, and the first lateral spectrum is brighter than any of the succeeding ones.

56. In practical applications the angle θ is so small, that θ may be written for $\sin\theta$; whence the angular distance between the centres of two bright bands is equal to $\lambda/(a+d)$. The breadths of the bands are therefore inversely proportional to $a+d$, and will therefore be broad when $a+d$ is very small; accordingly a fine grating having a very large number of lines, produces broader bands than one having a smaller number. Since the breadths of the bands are directly proportional to λ, the bands will be broader for red light than for violet light; accordingly when sunlight is employed, the outer edges of the central band will be red, and the inner edges of the two adjacent lateral bands will be violet.

If the values of the quantities for red and violet light be denoted by the suffixes r and v,

$$\theta_r = m\lambda_r/(a+d), \quad \theta_v = m\lambda_v/(a+d);$$

whence $$\theta_r - \theta_v = m(\lambda_r - \lambda_v)/(a+d),$$

which gives the angular value of the dispersion for the two extreme colours in the mth spectrum. This result shows the importance of having the lines ruled very close together, so that the dispersion may be as large as possible.

In order that the spectra may not overlap, it is necessary that the value of θ_v for the $(m+1)$th spectrum should be greater than the values of θ_r for the mth; which requires that

$$(m+1)\lambda_r > m\lambda_r.$$

Since λ_r is nearly equal to $2\lambda_v$, overlapping will take place in the spectra of higher orders than the first.

Resolving Power of Gratings.

57. We must now consider the resolving power of gratings; and shall first sum the series (42). Let

$$2\pi t/\tau = \phi, \quad 2\pi a/\lambda \, . \sin\theta = \alpha, \quad 2\pi(a+d)/\lambda \, . \sin\theta = \chi;$$

then the first vertical line of (42) becomes

$$[\cos(\phi+\alpha+n\chi) - \cos(\phi+\alpha-\chi) \\ - \cos\{\phi+\alpha+(n+1)\chi\} + \cos(\phi+\alpha)]/4\sin^2\tfrac{1}{2}\chi,$$

which reduces to

$$\frac{\cos(\phi+\alpha+\frac{1}{2}n\chi)\sin\frac{1}{2}(n+1)\chi}{\sin\frac{1}{2}\chi},$$

whence the resultant displacement at P becomes

$$\frac{\lambda\sin\frac{1}{2}(n+1)\chi}{2\pi\sin\theta\sin\frac{1}{2}\chi}\{\cos(\phi+\frac{1}{2}n\chi)-\cos(\phi+\alpha+\frac{1}{2}n\chi)\}.$$

We have already shown, that the intensity will be a maximum when

$$(a+d)\sin\theta = m\lambda, \quad \text{or} \quad \chi = 2m\pi.$$

We shall now show, that the intensity will be zero when

$$(a+d)\sin\theta = \left(m+\frac{s}{n+1}\right)\lambda \quad \text{............(47)},$$

where $s=1, 2...n$. For in this case

$$\tfrac{1}{2}(n+1)\chi = \pi\{m(n+1)+s\},$$

whence $$\sin\tfrac{1}{2}(n+1)\chi = 0.$$

Now n is the number of *opaque* lines, and $n+1$ is the number of *transparent* lines on the grating; consequently between the directions determined by (43), which may be called the *principal maxima*, there will be a series of dark lines equal to the number of opaque lines of the grating, which will be separated by bright lines, which may be called secondary maxima. The principal maxima are however far the most distinct, and the secondary maxima are so faint that they may be left out of consideration.

Let us now suppose, that the incident light consists of a double line of light of wave-lengths λ and $\lambda+\delta\lambda$. On account of the difference of wave-length, the maxima and minima of the two superimposed spectra will not coincide; but the want of coincidence will not be capable of being detected, unless the principal maximum of the mth spectrum light of wave-length $\lambda+\delta\lambda$, coincides with the *first* minimum succeeding the mth principal maximum of light of wave-length λ. Whence by (43) and (47) we must have

$$(a+d)\sin\theta = m(\lambda+\delta\lambda) = \left(m+\frac{1}{n+1}\right)\lambda,$$

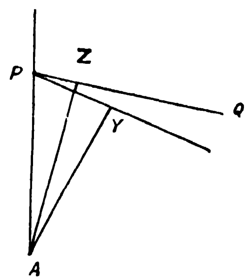
which gives $$\frac{\delta\lambda}{\lambda} = \frac{1}{m(n+1)} \text{........................(48)}.$$

This equation gives the smallest difference of wave-lengths in a double line which can just be resolved; consequently the resolving power of a grating depends solely upon the total number

of lines and the order of the spectrum. In the case of the D lines in the spectrum of sodium vapour, $\delta\lambda/\lambda = 1000^{-1}$, so that to resolve this line in the first spectrum requires a grating having 1000 (transparent) lines upon it; and in the second spectrum 500 lines, and so on. It is of course assumed in (48) that $n+1$ transparent lines are really utilized.

Reflection Gratings.

58. The gratings hitherto considered act by refraction; but it is possible to form a diffraction spectrum by means of a reflecting surface, on which a large number of fine lines are ruled. The fine lines act the part of the opaque obstacles in a refraction grating.



If YP be the incident and PZ the diffracted ray, and if i, ϕ be the angles which the incident and diffracted rays make with the normal to the grating, the disturbance at any point Q may be obtained by writing $\sin i + \sin\phi$ for $\sin\theta$ in (41). Hence the position of the mth spectrum is determined by

$$(a+d)(\sin i + \sin\phi) = m\lambda.$$

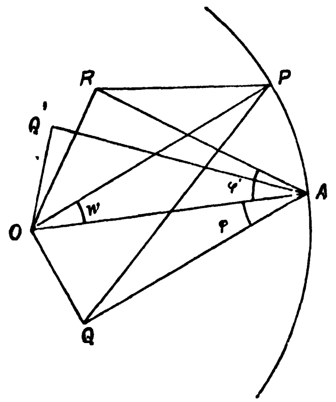
A similar formula holds good when light is incident obliquely upon a refraction grating.

Rowland's Concave Gratings[1].

59. In these gratings lines are ruled upon a concave spherical mirror made of speculum metal, and are the intersections of parallel planes one of which passes through the centre of the sphere.

[1] *Amer. Jour. of Science;* 3rd series, vol. XXVI. p. 87. Glazebrook, *Phil. Mag.* June and Nov. 1883.

In the figure let O be the centre of the mirror, Q a source of light, and let us for simplicity consider the state of things in the



plane QOA, so that the problem may be treated as one of two dimensions. Let Q' be the primary focus of Q; let $AQ = u$, $AQ' = v_1$, $QAO = \phi = Q'AO$; also let a be the radius of the mirror.

By geometrical optics

$$\frac{1}{v_1} + \frac{1}{u} = \frac{2 \sec \phi}{a};$$

hence if Q lie on the circle whose diameter is the radius of the mirror, $u = a \cos \phi$; whence $v_1 = a \cos \phi$, and therefore Q' also lies on the same circle.

The point Q' gives the position of the central diffraction band formed by the grating, but there will also be a series of lateral spectra arranged along this circle on either side of Q', which we shall now consider.

Let P be any point on the grating, let PR be the diffracted ray; also let $RAO = \phi'$, $POA = \omega$.

The retardation is $QP + PR - QA - AR$, which we must proceed to calculate. We have

$$QP^2 = u^2 + 4a^2 \sin^2 \tfrac{1}{2}\omega - 4au \sin \tfrac{1}{2}\omega \sin(\tfrac{1}{2}\omega - \phi)$$

$$= (u + a \sin \phi \sin \omega)^2 - a^2 \sin^2 \phi \sin^2 \omega + 4a(a - u \cos \phi) \sin^2 \tfrac{1}{2}\omega.$$

Now ω is usually a small quantity, whence if we neglect powers of $\sin \omega$ higher than the fourth, we may write

$$4 \sin^2 \tfrac{1}{2}\omega = \sin^2 \omega + \tfrac{1}{4} \sin^4 \omega,$$

whence

$$QP^2 = (u + a\sin\phi\sin\omega)^2 + a\cos\phi\sin^2\omega\,(a\cos\phi - u) + \tfrac{1}{4}\,a\,(a - u\cos\phi)\sin^4\omega\,;$$

accordingly if Q lie on the circle whose diameter is OA, so that $u = a\cos\phi$, the term involving $\sin^2\omega$ vanishes, and we obtain

$$QP = (u + a\sin\phi\sin\omega)\left\{1 + \frac{a\,(u - a\cos\phi)\sin^4\omega}{8\,(u + a\sin\phi\sin\omega)^2}\right\},$$

or $$QP - QA = a\sin\phi\sin\omega + \tfrac{1}{8}a\sin\phi\tan\phi\sin^4\omega.$$

If R also lie on the same circle, the value of $PR - AR$ will be obtained by writing $-\phi'$ for ϕ, whence

$$QP + PR - QA - AR$$
$$= a\,(\sin\phi - \sin\phi')\sin\omega + \tfrac{1}{8}a\,(\sin\phi\tan\phi + \sin\phi'\tan\phi')\sin^4\omega.$$

The advantage of this arrangement is, that the second term involves $\sin^4\omega$, and is therefore exceedingly small; the accuracy of the instrument is therefore far greater than one in which terms involving $\sin^2\omega$ and $\sin^3\omega$ occurred.

Neglecting the second term, it follows, from what has gone before, that the bright bands are given by the equation

$$\sigma\,(\sin\phi - \sin\phi') = \pm\, m\lambda.$$

where σ is the distance between two lines of the grating.

In order that a large part of the field of view may be in focus, the eye-piece is placed at O, whence $\phi' = O$ and

$$\sigma\sin\phi = \pm\, m\lambda,$$

and the different spectra can be observed by moving the source Q, (which is usually a narrow slit), along the circle OQA.

Michelson's Investigations.

60. The principle of Interference has been lately applied by Michelson to measure angular magnitudes of small sources of light. This was effected by providing the object-glass of a telescope with a cap, in which there are two slits adjustable in width and distance apart. If such a combination be focused upon a star, then instead of the image of the star, a series of coloured interference bands with a white centre will be observed, the bands being arranged at equal distances apart and parallel to the two slits. The position of the central white fringe can be marked from ten to fifty times as accurately, as the centre of the telescopic image of the star.

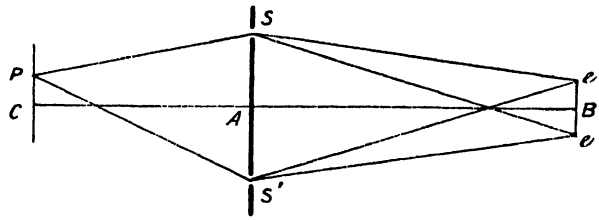
In the figure, let e, e' be two sources of light identical in every respect; let AB be the axis of the telescope, S, S' the slits in the cap. Let

$$ee' = a, \quad AB = d, \quad SS' = b \text{ .... } (49),$$

also let

$$\alpha = eAe', \quad \beta = SBS',$$

where α, β are very small angles.



Then the interference fringes produced at S and S' will vanish, when

$$Se' - Se = \tfrac{1}{2}\beta a = \tfrac{1}{2}b\alpha = \tfrac{1}{2}\lambda \text{ .... } (50),$$

that is when

$$\alpha = \lambda/b \text{ .... } (51).$$

But we have already shown, that λ/b is the limit of the resolving power of a telescope of aperture b, and if this quantity be denoted by α_0, (51) becomes

$$\alpha = \alpha_0.$$

Hence the fringes disappear, when the sources subtend an angle which can just be resolved by the telescope.

If however the source of light, instead of consisting of two distinct identical sources, consists of light coming from an object of finite size, it will be found *that on continuing to widen the slit the fringes again become clearly visible, and then reappear and disappear at regular intervals.*

61. The theory of these successive appearances and disappearances is as follows.

Let x be the distance of any element of the source from the axis of the telescope, dx the width of the element and $\phi(x)$ its length; and let us consider the intensity at a point P on a plane parallel to SS'. Let $AC = h$, $CP = y$, $Be = x$, $SCS' = \gamma$.

Then

$$S'e + S'P - Se - SP = bx/d + by/h$$
$$= \beta x + \gamma y;$$

whence the intensity at P due to the element $\phi(x)\,dx$ at e, is proportional to

$$\phi(x)\left\{1+\cos\frac{2\pi}{\lambda}(\beta x+\gamma y)\right\}dx.$$

The total intensity I^2 is obtained by integrating over the whole area of the source, whence

$$I^2=\int\phi(x)\left\{1+\cos\frac{2\pi}{\lambda}(\beta x+\gamma y)\right\}dx\,\ldots\ldots\ldots\,(52).$$

We shall now consider the case in which the source is a uniformly illuminated rectangle of width a, whose centre is in the axis of the telescope, and whose sides are parallel to SS'.

Putting $\phi(x)=1$, and integrating (52) between the limits $\frac{1}{2}a$ and $-\frac{1}{2}a$, we obtain

$$I^2=a+\frac{\lambda}{\pi\beta}\sin\frac{\pi\beta a}{\lambda}\cos\frac{2\pi\gamma y}{\lambda}\,\ldots\ldots\ldots\ldots\,(53).$$

If I_1^2, I_2^2 be the intensities at the centres of the bright and dark fringes respectively, Michelson assumes that the *visibility* V of the fringes is

$$V=\frac{I_1^2-I_2^2}{I_1^2+I_2^2}.$$

The values of I_1, I_2 are obtained by putting $y=n\lambda/\gamma$ and $(n+\frac{1}{2})\lambda/\gamma$ respectively, whence

$$V=\frac{\sin(\pi\beta a/\lambda)}{\pi\beta a/\lambda}\,\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\,(54).$$

We have shown in § 40, that the maxima values of this expression occur, when

$$\beta a/\lambda=1{\cdot}4303,\ 2{\cdot}4590,\ 3{\cdot}4709,\ \&c.$$

and the minima when

$$\beta a/\lambda=1,\ 2,\ 3\ldots\ldots$$

Since $\beta a=b\alpha$, it follows that if $\alpha=\lambda/b$, which is the limit of the resolving power, the fringes will be invisible; but if b, and consequently β, gradually increase, the fringes will become visible, and will again disappear when $\beta=2\lambda/a$. The fringes will therefore alternately appear and disappear as the distance between the slits increases.

A similar result takes place, when the width of the source increases, whilst the distance between the slits remains constant.

These results were experimentally verified by Michelson.

62. We shall now consider the case in which there are two identical sources of equal breadths, which are equidistant from B.

Let s be the distance of the centre of either source from B, $2r$ its breadth. Then the intensity due to the source whose centre is $x = s$, will be obtained by integrating (52) between the limits $s + r$ and $s - r$; and is therefore equal to

$$2r + \frac{\lambda}{\pi\beta} \cos \frac{2\pi}{\lambda} (\gamma y + \beta s) \sin \frac{2\pi\beta r}{\lambda}.$$

The intensity due to the other source is obtained by changing the sign of s, whence adding these two results, the total intensity is

$$I^2 = 4r + \frac{2\lambda}{\pi\beta} \cos \frac{2\pi\gamma y}{\lambda} \cos \frac{2\pi\beta s}{\lambda} \sin \frac{2\pi\beta r}{\lambda} \quad \ldots\ldots (55),$$

whence
$$V = \frac{\sin (2\pi\beta r/\lambda) \cos (2\pi\beta s/\lambda)}{2\pi\beta r/\lambda} \quad \ldots\ldots\ldots\ldots\ldots (56).$$

Let α be the angle subtended at A by the line joining the centres of the sources, then $2s/d = \alpha$, whence $2\pi\beta s/\lambda = \pi\alpha/\alpha_0$, where $\alpha_0 = \lambda/b$. Also let $2r/d = \alpha_1$, then $2\pi\beta r/\lambda = \pi\alpha_1/\alpha_0$; hence (56) becomes

$$V = \frac{\sin (\pi\alpha_1/\alpha_0) \cos (\pi\alpha/\alpha_0)}{\pi\alpha_1/\alpha_0} \quad \ldots\ldots\ldots\ldots\ldots (57).$$

By (54), the visibility due to a single source of breadth $2s$, is

$$V = \frac{\sin (\pi\alpha/\alpha_0)}{\pi\alpha/\alpha_0} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (58).$$

From these results we see that if a telescope is focused upon a double star, the fringes will be different from those produced by a single star; and the preceding investigation furnishes a method by means of which the double stars may be detected, which are incapable of being resolved by telescopes of the largest aperture in existence. Further information, together with the application of the theory to spectroscopic measurements, will be found in the papers by Michelson referred to below[1].

[1] "Measurement of Light Waves," *Amer. Jour. of Science*, vol. XXXIX. Feb. 1890. "Visibility of Interference Fringes in the Focus of a Telescope," *Phil. Mag.* March 1891, p. 256. "Application of Interference Methods to Spectroscopic Measurements," *Ibid.* April 1891, p. 338.

EXAMPLES.

1. Parallel homogeneous light from a source, is intercepted at right angles by a screen pierced with an aperture in the shape of a cross, consisting of two equal rectangles of sides $4a$, $2a$ superposed with their longest sides at right angles, and their centres coincident. Investigate the phenomena upon a distant screen.

2. A small luminous body is placed before a convex lens. A screen pierced with orifices of any form, stands between the luminous body and the lens. Show that the intensity of the image on a screen, placed at right angles to the axis of the lens, at the position where the image is formed, will be proportional to the sum of the areas of the orifices.

3. Plane waves of homogeneous light of wave-length λ impinge normally on a diaphragm, and are diffracted through an aperture in the diaphragm, and received on a parallel screen at a distance d. If the aperture be an annulus of radii a, A, prove that the intensity of light at a point on the screen, will be proportional to the square of

$$Ar^{-1}J_1(2\pi Ar/\lambda d) - ar^{-1}J_1(2\pi ar/\lambda d);$$

where r is the distance of the point from the projection of the centre of the annulus on the screen.

4. Homogeneous light of wave-length λ emanates from a point, and falls on a screen at a distance a from the point, in which there is a circular hole of radius r; the line joining the point to the centre of the hole being perpendicular to the screen. After passing through the hole, the light falls on a parallel screen at a distance b from the former. A circular ring of glass of thickness T, and refractive index μ and outer and inner radii r, r' is now placed in the hole. Find the change in the intensity of the illumination at the point on the screen opposite the centre of the hole, and show that this point will be black, provided $r = r'\sqrt{2}$, and

$$(\mu - 1)T = \tfrac{1}{2}\lambda(2n+1) - \tfrac{1}{4}(a+b)r^2/ab.$$

5. A screen is placed perpendicularly to the axis of a convex lens, and the image of a bright point Q on the axis is formed on the screen at q. If the light is allowed to pass through a small aperture, whose form is given by $y = \pm a \cos \pi x/2c$ from $x = -c$ to $x = c$, the origin being the centre C of the lens, examine the position of the dark points on the line through q parallel to Cx.

6. A, C, B, D are the middle points of the bounding edges of a rectangular diffraction grating, AB being parallel to the ruled lines, and being the central line of an opaque interval. The grating is blackened over except within the area $ACBD$. Find the system of fringes along the line which is the projection of AB upon the screen, and show how to find them for the projection of CD.

7. A plane vertical screen is one boundary of a semi-cylindrical dark chamber. A series of waves of light of given colour, the fronts of which are parallel to the screen, pass through it by a very narrow horizontal and rather short slit, of given length and equidistant from its vertical edges, and illuminate the opposite wall. Compare the brightness at different points of the wall in the horizontal plane through the slit, and prove that there is in this plane a succession of points of perfect darkness.

Prove that there is on the surface of the concave wall a series of dark bands, the projections of which on the screen are approximately a series of hyperbolas.

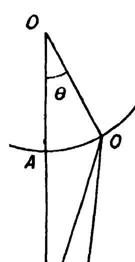
Supposing the colour of the slit to be changed, what alteration must be made in the length of the slit, in order that the position of the dark bands may not vary?

CHAPTER V.

DIFFRACTION CONTINUED.

63. WE have hitherto supposed that the waves are plane, or that they are converging to a focus; we shall now investigate the diffraction of light which is diverging from a focus.

We shall first suppose that the problem is one of two dimensions.



Let O be the focus, AQ the circle[1] representing the wave-front at the points of resolution; B any point at which the intensity is required. Let Q be any point on the wave-front, and let

$$OA = a, \quad AB = b, \quad AQ = s.$$

The vibration at B, produced by an element at Q may be taken to be proportional to

$$\cos 2\pi \left(\frac{t}{\tau} - \frac{\delta}{\lambda}\right) ds,$$

where $\delta = QB - AB$. Now

$$QB^2 = a^2 + (a+b)^2 - 2a(a+b)\cos s/a,$$

[1] For the lowest O in the figure, read Q.

whence $$QB = b + (a+b)\,s^2/2ab,$$
whence the vibration at P is

$$\int \cos 2\pi \left\{\frac{t}{\tau} - \frac{(a+b)\,s^2}{2\lambda ab}\right\} ds,$$

and the intensity I^2 is proportional to

$$I^2 = (\int \cos \tfrac{1}{2}\pi v^2 dv)^2 + (\int \sin \tfrac{1}{2}\pi v^2 dv)^2 \quad\text{............(1)},$$

where $$v^2 = 2\,(a+b)\,s^2/ab\lambda.$$

These integrals taken from 0 to v are called Fresnel's integrals, and we shall now discuss their properties.

Fresnel's Integrals.

64. The two definite integrals,

$$\int_0^v \cos \tfrac{1}{2}\pi v^2 dv \quad\text{and}\quad \int_0^v \sin \tfrac{1}{2}\pi v^2 dv,$$

cannot be evaluated in finite terms except in the single case in which $v = \infty$, when they are each equal to $\frac{1}{2}$. We shall denote them by C and S, and shall show how their values may be expressed in the form of a series.

By integration by parts, it follows that

$$\int_0^v v^{2n}\epsilon^{-a^2v^2} dv = \frac{v^{2n+1}\epsilon^{-a^2v^2}}{2n+1} + \frac{2a^2}{2n+1}\int_0^v v^{2n+2}\epsilon^{-a^2v^2} dv,$$

accordingly

$$\int_0^v \epsilon^{-a^2v^2} dv = \epsilon^{-a^2v^2}\left(v + \frac{2a^2v^3}{1.3} + \frac{2^2a^4v^5}{1.3.5} + \frac{2^3a^6v^7}{1.3.5.7} + \dots\right.$$

Putting $a = \frac{1}{2}(1+\iota)\,\pi^{\frac{1}{2}}$, so that $a^2 = \frac{1}{2}\iota\pi$, and equating the real and imaginary parts, we obtain

$$\left.\begin{aligned} C &= M\cos\tfrac{1}{2}\pi v^2 + N\sin\tfrac{1}{2}\pi v^2 \\ S &= M\sin\tfrac{1}{2}\pi v^2 - N\cos\tfrac{1}{2}\pi v^2 \end{aligned}\right\} \quad\text{..............(2)},$$

where

$$\left.\begin{aligned} M &= v - \frac{\pi^2v^5}{1.3.5} + \frac{\pi^4v^9}{1.3.5.7.9} \\ N &= \frac{\pi v^3}{1.3} - \frac{\pi^3v^7}{1.3.5.7} + \dots \end{aligned}\right\} \quad\text{........(3)}.$$

These two series are easily shown to be convergent, and are adapted for numerical calculation when v is small; they are due to Knockenhauer[1].

[1] *Die Undulationstheorie des Lichtes*, Berlin 1839, p. 36.

65. When v is large, the integrals may be expressed by means of certain semi-convergent series due to Cauchy[1]. We have

$$\int_0^v \epsilon^{-a^2v^2}\,dv = \frac{\sqrt{\pi}}{2a} - \int_v^\infty \epsilon^{-a^2v^2}\,dv.$$

Now $$\int_v^\infty \frac{\epsilon^{-a^2v^2}dv}{v^{2n}} = \frac{\epsilon^{-a^2v^2}}{2a^2v^{2n+1}} - \frac{2n+1}{2a^2}\int_v^\infty \frac{\epsilon^{-a^2v^2}dv}{v^{2n+2}},$$

whence

$$\int_v^\infty \epsilon^{-a^2v^2}\,dv = \epsilon^{-a^2v^2}\left(\frac{1}{2a^2v} - \frac{1}{2^2a^4v^3} + \frac{1\,.\,3}{2^3a^4v^5} - \frac{1\,.\,3\,.\,5}{2^4a^6v^7} + \dots\right).$$

Putting $a = \frac{1}{2}(1+\iota)\pi^{\frac{1}{2}}$ as before, and equating the real and imaginary parts, we obtain

$$\int_v^\infty \cos\tfrac{1}{2}\pi v^2 dv = -P\sin\tfrac{1}{2}\pi v^2 + Q\cos\tfrac{1}{2}\pi v^2,$$

$$\int_v^\infty \sin\tfrac{1}{2}\pi v^2 dv = \quad P\cos\tfrac{1}{2}\pi v^2 + Q\sin\tfrac{1}{2}\pi v^2,$$

where
$$\left.\begin{aligned} P &= \frac{1}{\pi v} - \frac{1\,.\,3}{\pi^3v^5} + \frac{1\,.\,3\,.\,5\,.\,7}{\pi^5v^9} - \dots \\ Q &= \frac{1}{\pi^2v^3} - \frac{1\,.\,3\,.\,5}{\pi^4v^7} + \frac{1\,.\,3\,.\,5\,.\,7\,.\,9}{\pi^6v^{11}} - \dots \end{aligned}\right\} \quad (4).$$

The first few terms of the series for P and Q converge rapidly when v is at all large, but the series ultimately become divergent. Such series are called semi-convergent series, and it is a known theorem that the sum of any number of terms of such a series differs from the true value of the function which the series represents, by a quantity less than the value of the last term included; we may therefore employ these series when v is large. Accordingly we find

$$\left.\begin{aligned} C &= \tfrac{1}{2} + P\sin\tfrac{1}{2}\pi v^2 - Q\cos\tfrac{1}{2}\pi v^2 \\ S &= \tfrac{1}{2} - P\cos\tfrac{1}{2}\pi v^2 - Q\sin\tfrac{1}{2}\pi v^2 \end{aligned}\right\} \dots\dots\dots\dots\dots (5).$$

66. Another method has been employed by Gilbert[2]. Put $u = \frac{1}{2}\pi v^2$, then

$$C = \frac{1}{(2\pi)^{\frac{1}{2}}}\int_0^u \frac{\cos u}{\sqrt{u}}\,du.$$

Now $$\frac{1}{\sqrt{u}} = \frac{1}{\sqrt{\pi}}\int_0^\infty \frac{\epsilon^{-ux}dx}{\sqrt{x}},$$

whence $$C = \frac{1}{\pi\sqrt{2}}\int_0^\infty dx\int_0^u \epsilon^{-ux}\cos u\,du.$$

[1] *C. R.* vol. xv. 534, 573.

[2] *Mém. couronnés de l'Acad. de Bruxelles*, vol. xxxi. 1.

The integration with respect to u can be effected, whence

$$C=\frac{1}{\pi\sqrt{2}}\left\{\int_0^\infty \frac{x^{\frac{1}{2}}dx}{1+x^2}-\cos u\int_0^\infty \frac{\epsilon^{-ux}x^{\frac{1}{2}}dx}{1+x^2}+\sin u\int_0^\infty \frac{\epsilon^{-ux}dx}{x^{\frac{1}{2}}(1+x^2)}\right\}.$$

Putting $x^2=y$, the first integral by a known formula[1] is equal to

$$\tfrac{1}{2}\pi\operatorname{cosec}\tfrac{3}{4}\pi=2^{-\frac{1}{2}}\pi;$$

we thus obtain $\qquad C=\frac{1}{2}-G\cos u+H\sin u \quad\text{..................(6)},$

where

$$\left.\begin{aligned}G&=\frac{1}{\pi\sqrt{2}}\int_0^\infty \frac{\epsilon^{-ux}x^{\frac{1}{2}}dx}{1+x^2}\\ H&=\frac{1}{\pi\sqrt{2}}\int_0^\infty \frac{\epsilon^{-ux}\,dx}{x^{\frac{1}{2}}(1+x^2)}\end{aligned}\right\}\quad\text{.(7)}.$$

By proceeding in a similar way, it can be shown that

$$S=\tfrac{1}{2}-G\sin u-H\cos u \quad\text{..................(8)}.$$

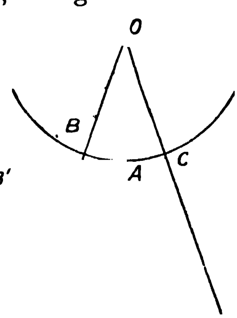
When $u=0$, it follows that

$$G=H=\tfrac{1}{2}.$$

Now u, which is equal to $\frac{1}{2}\pi v^2$, is necessarily positive, and consequently the factor ϵ^{-ux} decreases rapidly as u increases. It therefore follows that the values of G and H are never greater than $\frac{1}{2}$, and converge towards zero as u increases indefinitely. This property constitutes the superiority of Gilbert's method.

Straight Edge.

67. The first problem we shall consider will be that of diffraction by a straight edge.



First let the point P lie within the geometrical shadow. Let BAC be the front of the wave, and let $BA=h$; then the

[1] $\displaystyle\int_0^\infty \frac{y^{k-1}\,dy}{1+y}=\frac{\pi}{\sin k\pi}$, where $1>k>0$.

intensity at P is obtained by integrating from $s=h$ to $s=\infty$, whence if

$$V=\left\{\frac{2(a+b)}{ab\lambda}\right\}^{\frac{1}{2}}h,$$

the intensity will be proportional to

$$I^2=\left(\int_v^\infty \cos\tfrac{1}{2}\pi v^2 dv\right)^2+\left(\int_v^\infty \sin\tfrac{1}{2}\pi v^2 dv\right)^2$$

$$=(\tfrac{1}{2}-C_r)^2+(\tfrac{1}{2}-S_r)^2.$$

At the edge of the geometrical shadow, $h=0$, in which case $C=S=0$, and $I^2=\frac{1}{2}$; but when V is large, which will always be the case unless h is so small as to be comparable with $\lambda^{\frac{1}{2}}$, it follows from (4) that the most important term in I^2 is equal to $1/\pi^2V^2$. Whence the intensity at the edge of the geometrical shadow is proportional to $\frac{1}{2}$, and rapidly diminishes as P proceeds inwards.

For a point Q outside the shadow, we must integrate from $s=-h$ to $s=\infty$, where h is now equal to AC, whence

$$I^2=(\tfrac{1}{2}+C_r)^2+(\tfrac{1}{2}+S_r)^2.$$

The maxima and minima values of this expression are determined by the equation

$$(\tfrac{1}{2}+C_r)\frac{dC_r}{dV}+(\tfrac{1}{2}+S_r)\frac{dS_r}{dV}=0,$$

or $$(\tfrac{1}{2}+C_r)\cos\tfrac{1}{2}\pi V^2+(\tfrac{1}{2}+S_r)\sin\tfrac{1}{2}V^2=0.$$

When $V=0$ this expression is equal to $\frac{1}{2}$ and is therefore positive, and the corresponding value of the intensity is equal to $\frac{1}{2}$. Using the series (2) and (3), we see that when $V=1$, the value of this expression is equal to $\frac{1}{2}+S_r$ or $\frac{1}{2}+M$, which is also positive; but if $V^2=\frac{3}{2}$, the expression is equal to $-C_r+S_r$ which in this case is equal to $-2N$ if we employ (2) and (3), or equal to $-2Q$ if we employ (4) and (5). The expression in question is therefore negative when $V^2=\frac{3}{2}$, and therefore vanishes and changes sign for some value of V^2 lying between 1 and $\frac{3}{2}$. This root corresponds to a maximum value of the intensity. The first maximum according to Verdet[1] occurs when

$$\tfrac{1}{4}V^2=\tfrac{3}{8}-{\cdot}0046,$$

and the first minimum when

$$\tfrac{1}{4}V^2=\tfrac{7}{8}+{\cdot}0016.$$

[1] *Leçons d'Optique Physique*, vol. I. § 90.

The maxima and minima values have also been calculated by Fresnel, and are shown in the following table:

	$\dot{V}$	I^2
1st max.	1·2172	2·7413
„ min.	1·8726	1·5570
2nd max.	2·3449	2·3990
„ min.	2·7392	1·6867
3rd max.	3·0820	2·3022
„ min.	3·3913	1·7440

The effect of the screen is therefore to produce a series of bright bands outside the geometrical shadow of the source; whilst inside the shadow, there are no bands, but the intensity diminishes rapidly to zero.

Circular Aperture or Disc.

68. We have already considered the case of diffraction through a circular aperture, when light is converging to a focus; we shall now discuss the corresponding problem when light is diverging from a focus.

In the figure to § 63, let B be the projection of the centre of the aperture, AQ the wave-front at the aperture, Q any point on it; and let P be any point on the second screen, which is supposed to be parallel to the first screen in which the aperture exists.

Let $OA = a$, $AB = b$, $BP = r$, ϕ the angle which the plane OAQ makes with the plane OBP; also let c be the radius of the aperture.

Then

$$PQ^2 = (r - a\sin\theta\cos\phi)^2 + a^2\sin^2\theta\sin^2\phi + (a + b - a\cos\theta)^2$$
$$= b^2 + r^2 - 2ra\sin\theta\cos\phi + 4a(a+b)\sin^2\tfrac{1}{2}\theta.$$

Putting $\rho = a\sin\theta$, and treating ρ as small, we obtain

$$PQ = b + \frac{r^2}{2b} - \frac{r\rho\cos\phi}{b} + \frac{a+b}{2ab}\rho^2.$$

If therefore the vibration at A be denoted by $a^{-1}\cos 2\pi t/\tau$, the vibration at P will be expressed by

$$-\frac{1}{ab\lambda}\iint \sin 2\pi\left(\frac{t}{\tau}-\frac{PQ}{\lambda}\right)dS.$$

Now $dS=\rho d\rho d\phi$ very approximately; if therefore we put

$$\frac{\pi(a+b)}{ab\lambda}=\tfrac{1}{2}\kappa,\qquad \frac{2\pi r}{\lambda b}=l\ldots\ldots\ldots\ldots\ldots\ldots(9),$$

we shall find that the intensity is

$$I^2=\frac{1}{a^2b^2\lambda^2}(C^2+S^2)\ldots\ldots\ldots\ldots\ldots\ldots\ldots(10),$$

where

$$\left.\begin{aligned}C&=\iint\cos\left(\tfrac{1}{2}\kappa\rho^2-l\rho\cos\phi\right)\rho d\rho d\phi\\ S&=\iint\sin\left(\tfrac{1}{2}\kappa\rho^2-l\rho\cos\phi\right)\rho d\rho d\phi\end{aligned}\right\}\ldots\ldots\ldots\ldots(11).$$

The above expression for the intensity is of a perfectly general character, and the limits of integration must be chosen so as to include the whole of the aperture. When the aperture is a circle whose centre is A, the limits are from $\phi=0$ to 2π, and from $\rho=0$ to c; if on the other hand, we are investigating diffraction produced by a circular disc, the limits of ρ will be c and ∞.

69. Before discussing the general formulæ (10) and (11), we shall consider two special cases.

(i) Let the diffraction be produced by an aperture of radius c; then at the projection of its centre upon the screen, $r=l=0$, and

$$C=2\pi\int_0^c\cos\tfrac{1}{2}\kappa\rho^2.\rho d\rho=\frac{2\pi}{\kappa}\sin\tfrac{1}{2}\kappa c^2,$$

$$S=\frac{2\pi}{\kappa}(1-\cos\tfrac{1}{2}\kappa c^2),$$

whence

$$I^2=\frac{4}{(a+b)^2}\sin^2\tfrac{1}{4}\kappa c^2.$$

(ii) Let the diffraction be produced by a disc, then

$$\int_c^\infty x\epsilon^{-px^2}dx=\frac{1}{2p}\epsilon^{-pc^2},$$

whence if $p=\tfrac{1}{2}\iota\kappa$,

$$\int_c^\infty x\cos\tfrac{1}{2}\kappa x^2dx=-\frac{2\pi}{\kappa}\sin\tfrac{1}{2}\kappa c^2,$$

$$\int_c^\infty x\sin\tfrac{1}{2}\kappa x^2dx=\frac{2\pi}{\kappa}\cos\tfrac{1}{2}\kappa c^2,$$

whence

$$I^2=\frac{4\pi^2}{a^2b^2\lambda^2\kappa^2}=\frac{1}{(a+b)^2};$$

and therefore the intensity is the same as if the wave had passed on undisturbed.

In the middle of the eighteenth century, Delisle[1] had observed the existence of a bright spot at the centre of the shadow of a small circular disc; but this experiment had been so completely forgotten, that Poisson[2] brought forward the objection to Fresnel's theory, that it required the intensity at the centre of the shadow to be the same as if the wave had passed on undisturbed. The experiment was accordingly repeated by Arago[3], with a small disc whose diameter was one millimetre, and the required phenomenon was immediately observed.

70. The preceding results, which are of a fairly elementary character, are applicable only to the centre of the projection of the disc or aperture. The intensity at excentric points forms the subject of a very elaborate investigation by Lommel[4], which we shall proceed to consider.

In the case of a circular aperture or disc

$$C = 2\int_0^\pi\int \cos\left(\tfrac{1}{2}\kappa\rho^2 - l\rho\cos\phi\right)\rho\, d\rho\, d\phi$$

$$= 2\int_0\int \cos\left(\tfrac{1}{2}\kappa\rho^2\right)\cos\left(l\rho\cos\phi\right)\rho\, d\rho\, d\phi$$

$$= 2\pi\int J_0(l\rho)\cos\left(\tfrac{1}{2}\kappa\rho^2\right)\rho\, d\rho \quad\ldots\ldots\ldots\ldots\ldots\ldots (12).$$

Similarly $$S = 2\pi\int J_0(l\rho)\sin\left(\tfrac{1}{2}\kappa\rho^2\right)\rho\, d\rho \quad\ldots\ldots\ldots\ldots\ldots (13).$$

Except in the special case of $l=0$, the integrals (12) and (13) cannot be evaluated unless the limits are infinity and zero. We shall therefore first obtain their values in this case.

Let $$u = \int_0^\infty x\epsilon^{-a^2x^2} J_0(bx)\, dx;$$

then by a known formula[5]

$$J_0(bx) = \frac{2}{\pi}\int_0^\infty \sin\left(bx\cosh\phi\right) d\phi.$$

[1] *Mém. de l'anc. Acad. des Sciences*, 1715, p. 166.

[2] Verdet, *Leçons d'Opt. Phys.* vol. I. § 66.

[3] *Œuvres Complètes*, vol. VII. p. 1.

[4] *Abh. der II. Cl. der Kön. Bayer. Akad. der Wiss.* vol. XV. p. 233. In this paper, and also in another by the same author in the same volume, a large amount of interesting information concerning Bessel's functions will be found.

[5] A proof will be found in *Proc. Camb. Phil. Soc.* vol. V. p. 431.

But $$\int_0^\infty x\epsilon^{-a^2x^2}\sin bx\,dx = \frac{b\sqrt{\pi}}{4a^3}\,\epsilon^{-\frac{b^2}{4a^2}},$$

whence $$u = \frac{2}{\pi}\int_0^\infty\int_0^\infty x\epsilon^{-a^2x^2}\sin(bx\cosh\phi)\,d\phi\,dx$$

$$= \frac{b}{2a^3\sqrt{\pi}}\int_0^\infty \epsilon^{-\frac{b^2}{4a^2}\cosh^2\phi}\cosh\phi\,d\phi$$

$$= \frac{1}{2a^2}\,\epsilon^{-\frac{b^2}{4a^2}}.$$

Putting $a(1+\iota)/2^{\frac{1}{2}}$ for a, equating the real and imaginary parts, and then writing $b = l$, $a^2 = \frac{1}{2}\kappa$, we obtain

$$C_\infty = \frac{2\pi}{\kappa}\sin\frac{l^2}{2\kappa}, \qquad S_\infty = \frac{2\pi}{\kappa}\cos\frac{l^2}{2\kappa} \text{ ....(14).}$$

71. We shall next show, how series may be obtained by which the integrals C and S may be calculated, when the limits are c and 0.

It is known that $$2J_n' = J_{n-1} - J_{n+1},$$

$$\frac{2nJ_n}{z} = J_{n-1} + J_{n+1},$$

from which we deduce

$$z^nJ_{n-1} = \frac{d}{dz}(z^nJ_n) \text{ .... (15),}$$

$$-\frac{J_{n+1}}{z^n} = \frac{d}{dz}\left(\frac{J_n}{z^n}\right) \text{ .... (16).}$$

Using (15) and integrating by parts, we obtain

$$\int_0^c \rho^{n+1}J_n(l\rho)\,\epsilon^{\frac{1}{2}\iota\kappa\rho^2}\,d\rho = \frac{c^{n+1}}{l}J_{n+1}(lc)\,\epsilon^{\frac{1}{2}\iota\kappa c^2} - \frac{\iota\kappa}{l}\int_0^c \rho^{n+2}J_{n+1}(l\rho)\,\epsilon^{\frac{1}{2}\iota\kappa\rho^2}\,d\rho.$$

Accordingly

$$C + \iota S = \frac{2\pi c}{l}\,\epsilon^{\frac{1}{2}\iota\kappa c^2}\left(J_1 - \frac{\iota\kappa c}{l}J_2 + \frac{\iota^2\kappa^2c^2}{l^2}J_3 - \dots\right).$$

Writing $$\kappa c^2 = y, \quad lc = z \text{ .... (17),}$$

and equating the real and imaginary parts, we obtain

$$\left.\begin{aligned} C &= \pi c^2\left(\frac{\cos\frac{1}{2}y}{\frac{1}{2}y}U_1 + \frac{\sin\frac{1}{2}y}{\frac{1}{2}y}U_2\right) \\ S &= \pi c^2\left(\frac{\sin\frac{1}{2}y}{\frac{1}{2}y}U_1 - \frac{\cos\frac{1}{2}y}{\frac{1}{2}y}U_2\right) \end{aligned}\right\} \text{ .... (18),}$$

where
$$\left.\begin{aligned} U_1 &= \frac{y}{z} J_1(z) - \frac{y^3}{z^3} J_3(z) + \frac{y^5}{z^5} J_5(z) - \dots \\ U_2 &= \frac{y^2}{z^2} J_2(z) - \frac{y^4}{z^4} J_4(z) + \frac{y^6}{z^6} J_6(z) - \dots \end{aligned}\right\} \quad (19).$$

These series for U_1, U_2 are convenient for numerical calculation when y/z or $\kappa c/l$ is small.

72. Series which are suitable for calculation when y/z is large may be obtained as follows. By (16) we obtain

$$\int_0^c \frac{J_n(l\rho)}{\rho^{n-1}} \epsilon^{\frac{1}{2}\iota\kappa\rho^2} d\rho = \frac{J_n(lc)}{\iota\kappa c^n} \epsilon^{\frac{1}{2}\iota\kappa c^2} - \frac{l^n}{\iota\kappa 2^n n!} + \frac{l}{\iota\kappa} \int_0^c \frac{J_{n+1}(l\rho)}{\rho^n} \epsilon^{\frac{1}{2}\iota\kappa\rho^2} d\rho,$$

whence

$$C + \iota S = \frac{2\pi}{\iota\kappa}\left(J_0 + \frac{l}{\iota\kappa c} J_1 + \frac{l^2}{\iota^2\kappa^2 c^2} J_2 + \dots\right) \cdot \frac{\epsilon^{-\frac{1}{2}\iota l^2/\kappa}}{\iota\kappa}$$

accordingly equating the real and imaginary parts, we obtain

$$\left.\begin{aligned} C &= \pi c^2 \left(\frac{2}{y} \sin \frac{z^2}{2y} + \frac{\sin \frac{1}{2}y}{\frac{1}{2}y} V_0 - \frac{\cos \frac{1}{2}y}{\frac{1}{2}y} V_1 \right. \\ S &= \pi c^2 \left(\frac{2}{y} \cos \frac{z^2}{2y} - \frac{\cos \frac{1}{2}y}{\frac{1}{2}y} V_0 - \frac{\sin \frac{1}{2}y}{\frac{1}{2}y} V_1\right) \end{aligned}\right\} \quad .(20),$$

where
$$\left.\begin{aligned} V_0 &= J_0(z) - \frac{z^2}{y^2} J_2(z) + \frac{z^4}{y^4} J_4(z) - \dots \\ V_1 &= \frac{z}{y} J_1(z) - \frac{z^3}{y^3} J_3(z) + \frac{z^5}{y^5} J_5(z) - \dots \end{aligned}\right\} \quad .(21).$$

By (9) and (17) it follows that when $y = z$, $(a+b)/a = r/c$; this value of r corresponds to the edge of the geometrical shadow. Under these circumstances we have[1]

$$U_1 = V_1 = \tfrac{1}{2}\{J_0(z) + \cos z\}, \quad U_2 = V_0 = \tfrac{1}{2}\sin z.$$

From (18) and (20) we also find

$$\left.\begin{aligned} C\cos\tfrac{1}{2}y + S\sin\tfrac{1}{2}y &= \frac{2\pi c^2}{y} U_1 = \frac{2\pi c^2}{y}\left\{\sin\left(\frac{z^2}{2y} + \tfrac{1}{2}y\right) - V_1\right. \\ C\sin\tfrac{1}{2}y - S\cos\tfrac{1}{2}y &= \frac{2\pi c^2}{y} U_2 = -\frac{2\pi c^2}{y}\left\{\cos\left(\frac{z^2}{2y} + \tfrac{1}{2}y\right) - \right. \end{aligned}\right\} \dots (22),$$

therefore
$$\left.\begin{aligned} U_1 + V_1 &= \sin\left(\frac{z^2}{2y} + \tfrac{1}{2}y\right) \\ V_0 - U_2 &= \cos\left(\frac{z^2}{2y} + \tfrac{1}{2}y\right) \end{aligned}\right\} \quad (23).$$

[1] See Todhunter's *Functions of Laplace*, p. 320.

73. The functions U_1, U_2, V_0, V_1, although expressed in the form of infinite series, are of course finite quantities. In fact since the greatest values of J_0, $\cos\frac{1}{2}\kappa\rho^2$, and $\sin\frac{1}{2}\kappa\rho^2$ are unity, it follows that C and S cannot be greater than πc^2; and from (22) we see at once that U_1, U_2 cannot exceed certain limiting values; hence in calculating the values of these quantities, we may use whichever of the pairs of series (19) or (21) is most convenient. If we employ the series (21) we shall obtain the values of V_0, V_1, and the values of U_1, U_2 can then be found from (23).

Since $\qquad y = 2\pi (a+b) c^2/ab\lambda, \qquad z = 2\pi rc/\lambda b,$

it follows that z depends upon the position of the point at which the intensity is to be examined, but y does not. We must accordingly assign a definite value or series of values to y, and then calculate the values of U_1, U_2 corresponding to each of these definite values, for different values of r or z. This has been done by Lommel for a series of values from $y=\pi$ to $y=10\pi$, for the functions $2U_1/y$ and $2U_2/y$, and from these tables the values of the intensity can be calculated in the different cases which arise.

74. We shall now show how the intensity may be calculated in the case of diffraction through a circular aperture.

Since $y/z = (a+b)c/ar$, it follows that y/z is $>$ or $<$ 1, according as the point lies in the bright part of the screen or in the geometrical shadow; at points on the edge of the shadow, $y=z$.

By (18) the intensity is

$$I^2 = \frac{\pi^2 c^4}{a^2 b^2 \lambda^2}(U_1^2 + U_2^2).$$

The points of maxima and minima intensity are given by the equation

$$U_1 \frac{dU_1}{dz} + U_2 \frac{dU_2}{dz} = 0 \qquad\qquad (24).$$

From (15) and (19) we see that

$$\frac{dU_1}{dz} = -\frac{z}{y} U_2, \quad \frac{dU_2}{dz} = -J_1 + \frac{z}{y} U_2,$$

whence (24) reduces to

$$J_1 U_2 = 0.$$

The values of the roots of the equation $J_1(z/\pi) = 0$ have been calculated by Stokes[1], and the values of the roots of the equation

[1] *Trans. Camb. Phil. Soc.* vol. IX. p. 166; see *ante*, p. 55.

$U_2 = 0$ for $y = \pi, 2\pi, \ldots 10\pi$ have been calculated by Lommel. The following table gives the results for $y = \pi$ and $y = 5\pi$.

	z	$J_1(z)$	$2U_2/y$	I^2
$y = \pi.$	3·8317	0	+ ·1062	·0263 min.
	4·7154	...	0	·0320 max.
	7·0156	0	− ·0406	·0018 min.
	8·3060	...	0	·0055 max.
	10·1735	0	+ ·0162	·0003 min.
	11·5785	...	0	·0019 max.
$y = 5\pi.$	3·0308	...	0	·0130 min.
	3·6258	...	0	·0131 max.
	3·8317	0	+ ·0045	·0131 min.
	7·0156	0	+ ·1736	·0302 max.
	9·4407	...	0	·0181 min.
	10·1735	0	− ·0674	·0185 max.

From this table we see that the roots of the equation $J_1 = 0$ correspond to the minima, and those of the equation $U_2 = 0$ to the maxima values of the intensity. Since $\pi = 3·1416$, it follows that when $y = \pi$ the first minimum lies within the geometrical shadow; hence the intensity gradually diminishes from the projection of the centre of the aperture, until a point is reached somewhat beyond the edge of the geometrical shadow, where it attains a minimum value. As the distance increases, a series of alternate bright and dark rings become visible, until the intensity gradually fades away to zero.

The change in the value of y from π to 5π may be effected, either by increasing the radius of the aperture or by diminishing a or b; but whichever course is adopted, we see from the table that when $y = 5\pi$, a considerable number of maxima and minima fall within the luminous area. As we proceed further from the edge, it will be seen that the differences between the maxima and minima are greater than when $y = \pi$, and consequently in this case the rings are more distinct than in the former case.

75. When diffraction is produced by a circular disc of radius c, the integrals must be taken from ∞ to c. Now

$$2\pi \int_c^\infty J_0(l\rho) \cos(\tfrac{1}{2}\kappa\rho^2)\,\rho d\rho = C_\infty - C,$$

and
$$2\pi \int_c^\infty J_0(l\rho) \sin(\tfrac{1}{2}\kappa\rho^2)\,\rho d\rho = S_\infty - S,$$

where the values of C_∞, S_∞ are given by (14); whence

$$I^2 = \frac{1}{a^2b^2\lambda^2}\{(C_\infty - C)^2 + (S_\infty - S)^2\}.$$

By (14) and (20),

$$C_\infty - C = \frac{2\pi c^2}{y}(V_1 \cos \tfrac{1}{2}y - V_0 \sin \tfrac{1}{2}y),$$

$$S_\infty - S = \frac{2\pi c^2}{y}(V_1 \sin \tfrac{1}{2}y + V_0 \cos \tfrac{1}{2}y),$$

whence

$$I^2 = \frac{4\pi^2c^4}{a^2b^2y^2\lambda^2}(V_1^2 + V_0^2).$$

The maxima and minima values of I^2 are determined by the equation

$$V_0\frac{dV_0}{dz} + V_1\frac{dV_1}{dz} = 0. \qquad (25).$$

By (15) and (21) we see that

$$\frac{dV_0}{dz} = -J_1 - \frac{z}{y}V_1, \quad \frac{dV_1}{dz} = \frac{z}{y}V_0,$$

whence (25) becomes

$$V_0J_1 = 0,$$

from which it follows, that the maxima and minima are determined by the roots of the equations $V_0 = 0$, $J_1 = 0$. The roots of the former equation have been calculated by Lommel for the values $y = \pi, 2\pi, \ldots 8\pi$; and the results for $y = \pi$, $y = 4\pi$ are shown in the following table.

	z	$J_1(z)$	$2V_0/y$	I^2
$y=\pi$	2·1090	...	0	·0462 min.
	3·8317	0	− ·3527	·2259 max.
	4·4087	...	0	·2149 min.
	6·2389	...	0	·4933 max.
	7·0156	0	− ·6771	·4585 min.
	7·7069	...	0	·4862 max.
$y=4\pi$	2·3757	...	0	·0002 min.
	2·8317	0	− ·0697	·0049 max.
	5·4304	...	0	·0007 min.
	7·0156	0	+ ·0636	·0041 max.
	8·4383	...	0	·0017 min.
	10·1735	0	− ·0709	·0056 max.

From this table we see that when $y = \pi$, there is one minimum within the geometrical shadow; and that the exterior of the

shadow is surrounded by a series of rings. We also see that as z increases, the absolute value of the intensity rapidly increases, whilst the differences between its maxima and minima values diminish. The general appearance on the screen accordingly consists of a bright spot at the centre, surrounded by an obscure circular ring of varying intensity which exists in the neighbourhood of the geometrical shadow; and at more distant points the intensity is sensibly uniform.

When y is equal to 4π, the intensity within the geometrical shadow, and also the differences between the maxima and minima are very small. There is accordingly a well-defined shadow surrounding the central bright spot. As the boundary of the geometrical shadow is approached the intensity increases, and continues to increase rapidly after the boundary has been passed, until at some distance it becomes sensibly uniform.

It is a matter of common observation, that when the shadow of a well-defined object (such as the edge of a razor), is thrown upon a screen, the edge of the shadow is not sharply delineated, but an indistinct appearance is observed in its neighbourhood. This is accounted for by the foregoing theory, which shows that the intensity does not change abruptly to zero, but gradually diminishes, passing through a series of maxima and minima values.

On the Bessel's Function $J_{n+\frac{1}{2}}$.

76. Before proceeding to discuss Lommel's method of dealing with two-dimensional problems of diffraction, it will be convenient to make a short digression for the purpose of considering the Bessel's function $J_{n+\frac{1}{2}}$.

When n is an integer, it is known that $J_n(x)$ satisfies the equations

$$\frac{d^2J_n}{dx^2}+\frac{1}{x}\frac{dJ_n}{dx}+\left(1-\frac{n^2}{x^2}\right)J_n=0 \quad\text{...........(26),}$$

$$J'_n=\frac{n}{x}J_n-J_{n+1} \quad\text{.........................(27),}$$

$$J'_n=J_{n-1}-\frac{n}{x}J_n \quad\text{.........................(28),}$$

and the Bessel's function of order $n+\frac{1}{2}$, will be defined to be a function which satisfies these three equations, when $n+\frac{1}{2}$ is written for n, where n is any positive or negative integer.

Writing $n+\frac{1}{2}$ for n in (26), it may be written

$$\frac{d^2u}{dx^2}+\frac{1}{x}\frac{du}{dx}+\left\{1-\frac{(n+\frac{1}{2})^2}{x^2}\right\}u=0 \;\ldots\ldots\ldots\ldots(29),$$

whence if $n=\psi x^{n+\frac{1}{2}}$, we obtain

$$\frac{d^2\psi}{dx^2}+\frac{2(n+1)}{x}\frac{d\psi}{dx}+\psi=0 \;\ldots\ldots\ldots\ldots\ldots(30).$$

Equation (29), being of the second order, has two independent solutions. One solution may be expressed in the form of the series

$$\psi_n=1-\frac{x^2}{2.2n+3}+\frac{x^4}{2.4.(2n+3)(2n+5)}-\ldots.(31),$$

from which we see that $\psi_0=x^{-1}\sin x$. We shall therefore define the Bessel's function $J_{n+\frac{1}{2}}$, *where n is zero or any positive integer*, by the equation

$$J_{n+\frac{1}{2}}=\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\frac{x^{n+\frac{1}{2}}}{1.3\ldots(2n+1)}\left\{1-\frac{x^2}{2(2n+3)}+\frac{x^4}{2.4.(2n+3)(2n+5)}\right.$$
$$\left.-\frac{x^6}{2.4.6.(2n+3)(2n+5)(2n+7)}+\ldots\right\}\ldots(32)$$
$$=\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\frac{x^{n+\frac{1}{2}}}{1.3\ldots(2n+1)}\psi_n.$$

From the manner in which $J_{n+\frac{1}{2}}$ has been deduced, we see that it is a solution of (26); we have now to show that it satisfies (27) and (28). This can be at once done by substituting from (32) in the equations

$$\left.\begin{aligned}J'_{n+\frac{1}{2}}&=\frac{n+\frac{1}{2}}{x}J_{n+\frac{1}{2}}-J_{n+\frac{3}{2}}\\ J'_{n+\frac{1}{2}}&=J_{n-\frac{1}{2}}-\frac{(n+\frac{1}{2})}{x}J_{n+\frac{1}{2}}\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots(33),$$

which are what (27) and (28) become, when $n+\frac{1}{2}$ is written for n.

77. We must now consider the function $J_{-n-\frac{1}{2}}$, *where n is zero or any positive integer*. A series for ψ containing negative powers of n, can easily be shown to be

$$\psi_{-n-1}=\frac{1}{x^{2n+1}}\left\{1+\frac{x^2}{2(2n-1)}+\frac{x^4}{2.4.(2n-1)(2n-3)}\right.$$
$$\left.+\frac{x^6}{2.4.6(2n-1)(2n-3)(2n-5)}+\ldots\right\}\ldots(34),$$

from which we see that $\psi_{-1} = x^{-1}\cos x$. This series obviously represents a function different from (31), and therefore the series (31) and (34), each multiplied by an arbitrary constant, represent the complete solution of (30). If therefore we define the function $J_{-n-\frac{1}{2}}$ by the equation

$$J_{-n-\frac{1}{2}} = (-)^n \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{1.3\ldots(2n-1)}{x^{n+\frac{1}{2}}} \left\{1 + \frac{x^2}{2(2n-1)} + \frac{x^4}{2.4.(2n-1)(2n-3)} + \frac{x^6}{2.4.6(2n-1)(2n-3)(2n-5)} + \ldots\right\} \ldots\ldots(35),$$

it follows that $J_{-n-\frac{1}{2}}$ is a solution of (26); and by substituting from (35) in the equations

$$J'_{-n-\frac{1}{2}} = -\frac{n+\frac{1}{2}}{x} J_{-n-\frac{1}{2}} - J_{-n+\frac{1}{2}},$$

$$J'_{-n-\frac{1}{2}} = J_{-n-\frac{3}{2}} + \frac{n+\frac{1}{2}}{x} J_{-n-\frac{1}{2}},$$

which are what (27) and (28) become, when $-n-\frac{1}{2}$ is written for n, it is at once seen that they are satisfied.

78. It also follows from (32) and (35), that

$$J_{\frac{1}{2}} = \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} \sin x, \qquad J_{-\frac{1}{2}} = \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} \cos x \ \ldots\ldots\ldots(36).$$

If in (30) we write $y = x^2$, it can be shown that

$$\frac{d\psi_n}{dy} = \psi_{n+1},$$

accordingly $\quad \psi_n = \left(\frac{1}{2x}\frac{d}{dx}\right)^n \psi_0,$

whence $\quad J_{n+\frac{1}{2}} = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{x^{n+\frac{1}{2}}}{1.3\ldots(2n+1)} \left(\frac{1}{2x}\frac{d}{dx}\right)^n \frac{\sin x}{x},$

$$J_{-n-\frac{1}{2}} = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{1.3\ldots(2n-1)}{x^{n+\frac{1}{2}}} \left(\frac{1}{2x}\frac{d}{dx}\right)^n \frac{\cos x}{x},$$

from which we see that $J_{-n-\frac{1}{2}}$ is zero when $x = \infty$, and infinite when $x = 0$.

79. We are now in a position to explain Lommel's method[1].

The light is supposed to diverge from a linear source O, and to be received on a screen. Let B be the projection of O on the

[1] *Abh. der II. Cl. der Kön. Bayer. Akad. der Wiss.* vol. xv. p. 531.

screen, then if in the figure to § 63, we put $\phi = 0$, $BP = x$, $AQ = \rho$, we may prove in precisely the same manner, that

$$PQ = b + \frac{x^2}{2b} - \frac{x\rho}{b} + \frac{(a+b)\rho^2}{2ab},$$

whence the intensity is proportional to $I^2 = C^2 + S^2$,

where
$$\left.\begin{aligned} C &= \int \cos\left(\tfrac{1}{2}\kappa\rho^2 - l\rho\right) d\rho \\ S &= \int \sin\left(\tfrac{1}{2}\kappa\rho^2 - l\rho\right) d\rho \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(37),$$

$$\tfrac{1}{2}\kappa = \frac{\pi(a+b)}{ab\lambda}, \qquad l = \frac{2\pi x}{b\lambda} \quad \ldots\ldots\ldots\ldots\ldots(38),$$

and the origin of ρ is the intersection of the line OB with the wave-front at the point A. The integration extends over the effective portion of the wave.

80. The two principal problems, which we shall have to consider, are diffraction through a slit, and diffraction by a long narrow rectangular obstacle.

When the slit or obstacle, whose breadth is supposed to be equal to $2c$, is parallel to the screen, and is symmetrically placed, so that its middle line is the intersection of the plane passing through the source and B, the integration will be from c to $-c$ in the case of a slit, and from ∞ to c, and $-\infty$ to $-c$ in the case of an obstacle.

81. When the integration is from c to $-c$, the odd parts of the integrals disappear, and we thus obtain

$$\left.\begin{aligned} C &= 2\int_0^c \cos\tfrac{1}{2}\kappa\rho^2 \cos l\rho d\rho \\ S &= 2\int_0^c \sin\tfrac{1}{2}\kappa\rho^2 \cos l\rho d\rho \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots(39).$$

The integrals (39) cannot be evaluated in finite terms unless $c = \infty$; in this case, it may be shown by writing $a = \alpha(1+\iota)/2^{\frac{1}{2}}$ in the integral

$$\int_0^\infty \epsilon^{-a^2x^2} \cos 2rx dx = \frac{\sqrt{\pi}}{2a}\epsilon^{-\frac{r^2}{a^2}},$$

that
$$\left.\begin{aligned} \int_0^\infty \cos\tfrac{1}{2}\kappa\rho^2 \cos l\rho d\rho &= \left(\frac{\pi}{2\kappa}\right)^{\frac{1}{2}} \cos\left(\frac{l^2}{2\kappa} - \tfrac{1}{4}\pi\right. \\ &= c\left(\frac{\pi}{2y}\right)^{\frac{1}{2}} \cos\left(\frac{l^2}{2y} - \tfrac{1}{4}\pi\right) \\ \int_0^\infty \sin\tfrac{1}{2}\kappa\rho^2 \cos l\rho d\rho &= -c\left(\frac{\pi}{2y}\right)^{\frac{1}{2}} \sin\left(\frac{l^2}{2y} - \tfrac{1}{4}\pi\right) \end{aligned}\right\} \ldots(40),$$

where
$$\kappa c^2 = y, \quad lc = z \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(41).$$

82. We shall now show how to express the values of C and S in series. Since

$$J_{-\frac{1}{2}}(x) = \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} \cos x,$$

it follows that

$$\left.\begin{aligned} C &= (2\pi)^{\frac{1}{2}} \int_0^c (l\rho)^{\frac{1}{2}} J_{-\frac{1}{2}}(l\rho) \cos \tfrac{1}{2}\kappa\rho^2 d\rho \\ S &= (2\pi)^{\frac{1}{2}} \int_0^c (l\rho)^{\frac{1}{2}} J_{-\frac{1}{2}}(l\rho) \sin \tfrac{1}{2}\kappa\rho^2 d\rho \end{aligned}\right\} \quad \text{.........(42).}$$

Now by (32) $J_{n+\frac{1}{2}}$ vanishes when $x = 0$, provided n is zero or any positive integer, whence integrating by parts, and using (15), we find

$$\int_0^c \rho^{n+\frac{1}{2}} J_{n-\frac{1}{2}}(l\rho) \epsilon^{\frac{1}{2}\iota\kappa\rho} d\rho = \frac{c^{n+\frac{1}{2}}}{l} J_{n+\frac{1}{2}}(l\rho) \epsilon^{\frac{1}{2}\iota\kappa c}$$
$$- \frac{\iota\kappa}{l} \int_0^c \rho^{n+\frac{3}{2}} J_{n+\frac{1}{2}}(l\rho) \epsilon^{\frac{1}{2}\iota\kappa\rho^2} d\rho,$$

whence

$$C + \iota S = \left(\frac{2\pi c}{l}\right)^{\frac{1}{2}} \epsilon^{\frac{1}{2}\iota\kappa c^2} \left\{J_{\frac{1}{2}} - \frac{\iota\kappa c}{l} J_{\frac{3}{2}} + \left(\frac{\iota\kappa c}{l}\right)^2 J_{\frac{5}{2}} - \ldots\right\}.$$

Equating the real and imaginary parts, and using (41), we obtain

$$\left.\begin{aligned} C &= \left(\frac{2\pi}{y}\right)^{\frac{1}{2}} c\,(U_{\frac{1}{2}} \cos \tfrac{1}{2}y + U_{\frac{3}{2}} \sin \tfrac{1}{2}y) \\ S &= \left(\frac{2\pi}{y}\right)^{\frac{1}{2}} c\,(U_{\frac{1}{2}} \sin \tfrac{1}{2}y - U_{\frac{3}{2}} \cos \tfrac{1}{2}y) \end{aligned}\right\} \quad \text{.........(43),}$$

where
$$U_n = \sum_0^{p=\infty} (-)^p \left(\frac{y}{z}\right)^{n+2p} J_{n+2p}(z) \quad \text{...........(44).}$$

83. To obtain a series in descending powers of z/y, we must recollect that $J_{-n-\frac{1}{2}}$ is zero when $x = \infty$, and infinite when $x = 0$. We must therefore write

$$C + \iota S = 2\int_0^\infty \epsilon^{\frac{1}{2}\iota\kappa\rho^2} \cos l\rho d\rho - (2\pi)^{\frac{1}{2}} \int_c^\infty (l\rho)^{\frac{1}{2}} J_{-\frac{1}{2}}(l\rho) \epsilon^{\frac{1}{2}\iota\kappa\rho^2} d\rho \ldots (45).$$

By integration by parts, and by (15), we can show that

$$\int_c^\infty \rho^{-n+\frac{1}{2}} J_{-n-\frac{1}{2}}(l\rho) \epsilon^{\frac{1}{2}\iota\kappa\rho^2} d\rho$$
$$= \frac{\iota J_{-n-\frac{1}{2}}(l\rho)}{\kappa c^{n+\frac{1}{2}}} + \frac{\iota l}{\kappa} \int_c^\infty \rho^{-n-\frac{1}{2}} J_{-n-\frac{3}{2}}(l\rho) \epsilon^{\frac{1}{2}\iota\kappa\rho^2} d\rho,$$

where n is zero or any positive integer; whence

$$(2\pi l)^{\frac{1}{2}} \int_c^\infty \rho^{\frac{1}{2}} \epsilon^{\frac{1}{2}\iota\kappa\rho^2} J_{-\frac{1}{2}}(l\rho)\, d\rho$$

$$= \iota\, (2\pi l)^{\frac{1}{2}} \epsilon^{\frac{1}{2}\iota\kappa c^2} \left\{ \frac{J_{-\frac{1}{2}}}{\kappa c^{\frac{1}{2}}} + \frac{\iota l}{\kappa^2 c^{\frac{3}{2}}} J_{-\frac{3}{2}} + \frac{\iota^2 l^2}{\kappa^3 c^{\frac{5}{2}}} J_{-\frac{5}{2}} + \ldots\ldots \right\},$$

whence equating the real and imaginary parts, we obtain

$$\left.\begin{aligned} &(2\pi l)^{\frac{1}{2}} \int_c^\infty \rho^{\frac{1}{2}} J_{-\frac{1}{2}}(l\rho) \cos \tfrac{1}{2}\kappa\rho^2 d\rho \\ &\qquad = -\left(\frac{2\pi}{y}\right)^{\frac{1}{2}} c\, (V_{\frac{1}{2}} \sin \tfrac{1}{2}y + V_{\frac{3}{2}} \cos \tfrac{1}{2}y) \\ &(2\pi l)^{\frac{1}{2}} \int_c^\infty \rho^{\frac{1}{2}} J_{-\frac{1}{2}}(l\rho) \sin \tfrac{1}{2}\kappa\rho^2 d\rho \\ &\qquad = \left(\frac{2\pi}{y}\right)^{\frac{1}{2}} c\, (V_{\frac{1}{2}} \cos \tfrac{1}{2}y - V_{\frac{3}{2}} \sin \tfrac{1}{2}y) \end{aligned}\right\} \ldots\ldots(46).$$

where $$V_n = \sum_{p=0}^{p=\infty} (-)^p \left(\frac{z}{y}\right)^{n+2p} J_{-n-2p}(z) \ \ldots\ldots\ldots\ldots(47).$$

If therefore we denote the right-hand sides of (46) by C', S' respectively, we have by (40) and (45),

$$\left.\begin{aligned} C &= \ \ c\left(\frac{2\pi}{y}\right)^{\frac{1}{2}} \cos\left(\frac{z^2}{2y} - \tfrac{1}{4}\pi\right) - C' \\ S &= -c\left(\frac{2\pi}{y}\right)^{\frac{1}{2}} \sin\left(\frac{z^2}{2y} - \tfrac{1}{4}\pi\right) - S' \end{aligned}\right\} \ldots\ldots\ldots\ldots(48).$$

Diffraction through a Slit.

84. We are now prepared to investigate the case of diffraction through a slit.

Since the intensity is proportional to

$$I^2 = C^2 + S^2,$$

it follows from (43) that

$$I^2 = \frac{2\pi c^2}{y} (U_{\frac{1}{2}}^2 + U_{\frac{3}{2}}^2),$$

and the maxima and minima values are given by

$$U_{\frac{1}{2}} \frac{dU_{\frac{1}{2}}}{dz} + U_{\frac{3}{2}} \frac{dU_{\frac{3}{2}}}{dz} = 0 \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots(49).$$

From (16) and (44) we obtain

$$\frac{dU_{\frac{1}{2}}}{dz} = -\frac{z}{y} U_{\frac{3}{2}}, \quad \frac{dU_{\frac{3}{2}}}{dz} = -\frac{z}{y} U_{\frac{5}{2}}$$

also from (44)

$$U_{\frac{1}{2}} + U_{\frac{5}{2}} = \left(\frac{y}{z}\right)^{\frac{1}{2}} J_{\frac{1}{2}}$$

whence (49) reduces to

$$z^{\frac{1}{2}} J_{\frac{1}{2}} U_{\frac{3}{2}} = 0.$$

Now $z^{\frac{1}{2}} J_{\frac{1}{2}} = (2/\pi)^{\frac{1}{2}} \sin z$, which vanishes when $z = n\pi$, where n is zero or any positive negative integer; accordingly there is a series of bands parallel to the edges of the slit, whose distances apart are equal to $\frac{1}{2} b\lambda/c$. Another system of bands is given by the roots of the equation $U_{\frac{3}{2}} = 0$.

Tables have been constructed by Lommel, which give the values of these roots, when $y = 3$, 6, 9...30, and the following table gives the results when $y = 3$ and $y = 15$.

	z	$J_{\frac{1}{2}}(z)$	$(\pi/2y)^{\frac{1}{2}} U_{\frac{3}{2}}$	I^2
$y = 3$	0	0	·7652	·8163 max.
	π	0	·1749	·0822 min.
	4·0127	...	0	·0949 max.
	2π	0	− ·0829	·0075 min.
	7·6130	...	0	·0223 max.
	3π	0	+ ·0366	·0014 min.
$y = 15$	0	0	·2012	·1244 max.
	π	0	·1546	·0813 min.
	2π	0	·3955	·1608 max.
	8·7546	...	0	·1132 min.
	3π	0	− ·1739	·1157 max.
	4π	0	− ·0610	·0498 min.

Since $y/z = (a + b)c/ax$, it follows that when $y > z$, the point x lies within the luminous area; and since $\pi = 3.1416$, it follows that when $y = 3$, the first minimum, and all subsequent maxima and minima lie in the geometrical shadow. On the other hand when $y = 15$, under which circumstances the slit is broader than when $y = 3$, there are a succession of maxima and minima within the luminous area.

Diffraction by a Narrow Obstacle.

85. We shall now suppose, that diffraction is produced by a long narrow rectangular obstacle of breadth $2c$.

In this case the intensity is proportional to

$$I^2 = C'^2 + S'^2,$$

where by (46)

$$C' = -c\left(\frac{2\pi}{y}\right)^{\frac{1}{2}}\left(V_{\frac{1}{2}}\sin\tfrac{1}{2}y + V_{\frac{3}{2}}\cos\tfrac{1}{2}y\right),$$

$$S' = c\left(\frac{2\pi}{y}\right)^{\frac{1}{2}}\left(V_{\frac{1}{2}}\cos\tfrac{1}{2}y - V_{\frac{3}{2}}\sin\tfrac{1}{2}y\right),$$

whence
$$I^2 = \frac{2\pi c^2}{y}\left(V_{\frac{1}{2}}^2 + V_{\frac{3}{2}}^2\right),$$

and the maxima and minima are determined by the equation

$$V_{\frac{1}{2}}\frac{dV_{\frac{1}{2}}}{dz} + V_{\frac{3}{2}}\frac{dV_{\frac{3}{2}}}{dz} = 0 \qquad\qquad (50).$$

By (15) and (47) we obtain

$$\frac{dV_{\frac{1}{2}}}{dz} = -\frac{z}{y}\Sigma(-)^p\left(\frac{z}{y}\right)^{-\frac{1}{2}+2p} J_{\frac{1}{2}-2p} = -\frac{z}{y}V_{-\frac{1}{2}},$$

and
$$\frac{dV_{\frac{3}{2}}}{dz} = -\frac{z}{y}V_{\frac{1}{2}},$$

also by (47)
$$V_{-\frac{1}{2}} + V_{\frac{3}{2}} = \left(\frac{y}{z}\right)^{\frac{1}{2}} J_{\frac{1}{2}},$$

whence (50) reduces to

$$z^{\frac{1}{2}} J_{\frac{1}{2}} V_{\frac{1}{2}} = 0.$$

In this case also there are a series of maxima or minima values corresponding to $z = 0, \pi, 2\pi\ldots$; whilst another set are given by the roots of the equation $V_{\frac{1}{2}} = 0$. The results are shown in the following table.

	z	$J_{\frac{1}{2}}$	$(\pi/2y)^{\frac{1}{2}} V_{\frac{1}{2}}$	I^2
$y=3$	0	0	$+\cdot2910$	·0891 max.
	1·3550	...	0	·0118 min.
	π	0	$-\cdot3350$	·1942 max.
	3·7710	...	0	·1769 min.
	5·7037	...	0	·6413 max.
	2π	0	$-\cdot6961$	·6130 min.
$y=12$	0	0	$+\cdot0819$	·0067 max.
	1·5426	...	0	·0001 min.
	π	0	$-\cdot0866$	·0076 max.
	4·6103	...	0	·0011 min.
	2π	0	$+\cdot1024$	·0108 max.
	7·6163	..	0	·0045 min.

When $y=3$, so that the obstacle is narrow, there is a central bright band, on either side of which are two bands of minimum intensity, which lie within the shadow; but when $y=12$, so that the obstacle is broader, there are several.

Diffraction by a Straight Edge.

86. The last problem, which we shall consider, is that of diffraction by an indefinitely large straight screen, which extends from the origin to infinity in the negative direction. In this case

$$\left.\begin{aligned} C &= \int_0^\infty \cos\left(\tfrac{1}{2}\kappa\rho^2 - l\rho\right) d\rho \\ S &= \int_0^\infty \sin\left(\tfrac{1}{2}\kappa\rho^2 - l\rho\right) d\rho \end{aligned}\right\} \quad \text{.(51).}$$

At the projection of the diffracting edge on the screen, $l=0$, and

$$C = S = \tfrac{1}{2}(\pi/\kappa)^{\frac{1}{2}},$$

and
$$I^2 = \tfrac{1}{2}\pi/\kappa.$$

If the diffracting edge were absent, the limits would be ∞ and $-\infty$, and we should have at *any* point of the screen

$$C = \int_{-\infty}^{\infty} \cos\tfrac{1}{2}\kappa\rho^2 \cos l\rho d\rho = 2\left(\frac{\pi}{2\kappa}\right)^{\frac{1}{2}} \sin\left(\frac{l^2}{2\kappa} + \tfrac{1}{4}\pi\right),$$

$$S = 2\left(\frac{\pi}{2\kappa}\right)^{\frac{1}{2}} \cos\left(\frac{l^2}{2\kappa} + \tfrac{1}{4}\pi\right).$$

Whence if I'^2 be the intensity,

$$I'^2 = 2\pi/\kappa,$$

and therefore $I^2 = \frac{1}{4}I'^2$; or the intensity at the edge of the geometrical shadow, is one quarter what it would be if the obstacle were removed.

When l is not zero, the integrals (51) cannot be evaluated in finite terms; they may however be reduced to Fresnel's integrals. We have

$$C = \int_0 (\cos \tfrac{1}{2}\kappa\rho^2 \cos l\rho + \sin \tfrac{1}{2}x\rho^2 \sin l\rho)\, d\rho$$

$$= \left(\frac{\pi}{2\kappa}\right)^{\frac{1}{2}} \sin\left(\frac{l^2}{2\kappa} + \tfrac{1}{4}\pi\right) + \int_0^\infty \sin \tfrac{1}{2}x\rho^2 \sin l\rho d\rho.$$

Let
$$u = \int_0^\infty \epsilon^{-a^2x^2} \sin 2bx dx,$$

then
$$\frac{du}{db} = 2\int_0^\infty x\epsilon^{-a^2x^2} \cos 2bx dx$$

$$= \frac{1}{a^2} - \frac{2bu}{a^2},$$

whence
$$u = \frac{\epsilon^{-\frac{b}{a^2}}}{a^2}\int_0^b \epsilon^{\frac{x^2}{a^2}}\, dx.$$

Writing $a = c(1+\iota)/2^{\frac{1}{2}}$, we obtain

$$\int_0^\infty \sin c^2x^2 \sin 2bx dx = \frac{1}{c^2}\int_0^b \cos \frac{b^2 - x^2}{c^2}\, dx,$$

$$\int_0^\infty \cos c^2x^2 \sin 2bx dx = \frac{1}{c^2}\int_0^b \sin \frac{b^2 - x^2}{c^2}\, dx.$$

The integrals on the right-hand side depend upon Fresnel's integrals, and accordingly C and S can be expressed by these quantities.

CHAPTER VI.

DOUBLE REFRACTION.

87. WE have already drawn attention to the fact, that there are certain crystalline substances, called doubly refracting crystals, which possess the property of separating a single ray of light into two rays. We shall now consider the experimental facts connected with this class of bodies.

One of the best examples of a doubly refracting crystal is a crystallized form of carbonate of lime called Iceland spar. Crystals of Iceland spar can easily be split into rhombohedra, the acute and obtuse angles of the faces of which, are equal to $74° 55' 35''$ and $105° 4' 25''$ respectively. The line joining the two opposite corners, where the obtuse angles meet, is called the optic axis of the crystal, and is a line with respect to which the properties of the crystal are symmetrical. Iceland spar therefore possesses the same kind of symmetry as an ellipsoid of revolution, and crystals of this class are called *uniaxal crystals*.

There are certain other kinds of doubly refracting crystals, which have two optic axes, and which possess three rectangular planes of symmetry. Such crystals are called *biaxal crystals*.

Uniaxal Crystals.

88. When a small pencil of light is incident upon a plate of uniaxal crystal, it is found that in general there are two refracted rays. One of these rays is refracted according to the ordinary law of refraction, and is consequently termed the *ordinary* ray; whilst the other is refracted according to a totally different law, and is called the *extraordinary* ray. There are however two cases in which there is only one refracted ray, viz. (i) when the direction of propagation coincides with the optic axis of the crystal, (ii) when the face of the crystal contains the axis, and the pencil is incident normally upon the surface. In both these cases the ordinary and extraordinary rays coincide, and only one refracted ray is consequently observed.

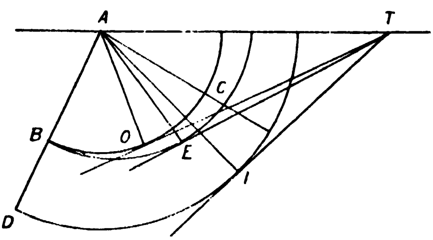
Let us now suppose, that a ray of light is refracted through a rhomb of Iceland spar, and that the plane of incidence contains the axis; and let the two refracted rays be transmitted through a second rhomb. When the two rhombs are similarly situated, it will be found that there are only two rays after refraction through the second rhomb, and that the ordinary ray O in the first rhomb, gives rise to an ordinary ray O_0 in the second, whilst the extraordinary ray E in the first rhomb, gives rise to an extraordinary ray E_E in the second. If now the second rhomb be turned through any angle which is less than 90°, it will be found that there are four refracted rays, and that O and E each give rise to an ordinary and an extraordinary ray O_0, O_E, and E_0, E_E respectively. When the angle is small, O_E and E_0 are very faint, but become brighter as the angle increases, whilst O_0 and E_E diminish in brightness; and when the second rhomb has been turned through an angle of 90°, O_0 and E_E will have disappeared, leaving O_E and E_0 in possession of the field.

These experimental results show, that doubly refracting crystals, in addition to dividing an incident ray into two refracted rays, also produce an essential modification in the constitution of the refracted light.

89. The index of refraction of the ordinary ray is the ratio of the sine of the angle of incidence to the angle of refraction, which as we have already seen is constant for all angles. The extraordinary index of refraction is defined as follows. Let the

plane of incidence contain the axis, and let a pencil of light be incident at such an angle, that the extraordinary ray is perpendicular to the axis; then the ratio of the angle of incidence to the angle of refraction of the extraordinary ray under these circumstances, is called the extraordinary index of refraction. The reason of this definition will appear hereafter; it can be proved at once by geometry, that this ratio is entirely independent of the inclination of the optic axis to the face of the crystal.

90. The law which determines the refraction of the extraordinary ray in uniaxal crystals was first discovered experimentally by Huygens[1], who gave the following construction.



Let A be the point of incidence, AB the direction of the optic axis; draw AC perpendicular to AB. Let B and D be points on AB, such that AD/AB is equal to the ordinary index of refraction, and let C be a point on AC such that AD/AC is equal to the extraordinary index. With A as a centre, describe two spheres whose radii are AB, AD; and describe also an ellipsoid of revolution, whose polar axis is equal to and coincident with AB, and whose equatorial axis is AC. Produce the incident ray to meet the second sphere in I, and let the tangent plane at I cut the surface of the crystal in a line T. Through T draw two tangent planes TO, TE to the first sphere and the ellipsoid respectively, meeting them in O and E; join AO, AE. Then AO, AE will be the directions of the ordinary and extraordinary rays respectively.

This construction was discovered by Huygens by a process of induction, but was afterwards verified by careful measurements.

91. The preceding construction suggests, that the wave-surface in a uniaxal crystal consists of two sheets, viz. a sphere

[1] *Traité de la Lumière.*

and an ellipsoid of revolution, which touch one another at the extremities of the optic axis; and we shall hereafter see that this conclusion is borne out both by theory and experiment. When the disturbance producing light is communicated to any point of the medium, two waves are generated, one of which is spherical and travels with the same velocity in all directions, whilst the other is spheroidal, and its velocity is different in different directions. When a plane wave is incident upon the surface of the crystal, each point of the surface may by Huygen's Principle, § 14, be regarded as the origin of secondary waves, and the envelop of these secondary waves will consist of two planes TO, TE, which are the fronts of the ordinary and extraordinary waves in the crystal. If the equations of the sphere and the ellipsoid of revolution, referred to A as origin, and AB as the axis of z, be

$$x^2 + y^2 + z^2 = c^2,$$

$$z^2/c^2 + (x^2 + y^2)/a^2 = 1,$$

then c will be the velocity of the ordinary wave, whilst the velocity of the extraordinary wave will be equal to the perpendicular drawn from A on to the wave-front TE. Since the extraordinary wave-front touches the ellipsoid, it follows that if θ be the angle which the direction of the wave makes with the optic axis, and V be its velocity of propagation,

$$V^2 = c^2 \cos^2 \theta + a^2 \sin^2 \theta.$$

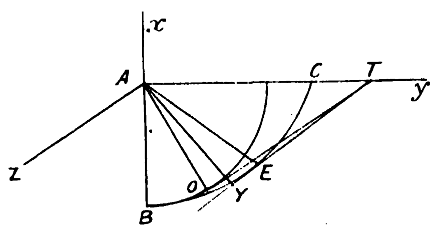
We therefore see that the directions of the two waves coincide, whenever they are parallel to the optic axis, or to an equatorial axis; in the former case $V = c$, and in the latter $V = a$. The quantities a and c are therefore called the principal wave velocities, and the ratios V/c, V/a are called the ordinary and extraordinary indices of refraction.

92. We have already pointed out, that when common light is incident upon a crystal, two refracted rays are always produced; on the other hand we have shown, that when the incident light consists of the ordinary or extraordinary ray, which is produced by refracting common light through another crystal, there are always two positions of the second crystal, in which one of the two rays is absent.

We shall now explain how this phenomenon may be accounted for.

93. We have stated in Chapter I., that light is said to be *polarized,* when the elements of ether composing the wave are vibrating perpendicularly to a fixed plane, which is called the plane of polarization. Now when common light is refracted through a crystalline plate, it is supposed that the two refracted rays are polarized in perpendicular planes; and that the vibrations of the ordinary ray are perpendicular to the plane, which passes through the optic axis and the normal to the ordinary wave-front; whilst the vibrations of the extraordinary ray lie in the plane passing through the optic axis and the normal to the extraordinary wave-front. The plane which passes through the optic axis and the normal to the wave-front is called the principal plane for that wave; we may therefore say, that the ordinary wave is polarized in the principal plane, whilst the extraordinary wave is polarized perpendicularly to the principal plane.

94. We are now able to explain why it is, that in certain positions of the crystal one of the two rays in certain cases disappears.



For simplicity, let the surface of the crystal be perpendicular to the optic axis AB; let xy be the plane of incidence, TO, TE be the ordinary and extraordinary wave-fronts, and AO, AE the ordinary and extraordinary rays.

When the incident light is polarized in the plane xy, the vibrations are parallel to Az. But since we have assumed, that the vibrations in the extraordinary wave are executed *in* the plane xy, it follows that an incident wave, whose vibrations are *perpendicular* to this plane, cannot give rise to an extraordinary wave, but only to an ordinary wave. When, on the other hand, the incident light is polarized *perpendicularly* to the plane xy, so that the vibrations are executed *in* that plane, the incident light gives rise to an extraordinary wave, but not to an ordinary wave. If the incident light were polarized in any other plane, the incident vibrations could be resolved into two components

respectively in and perpendicularly to the plane of incidence xy, the first of which would give rise to an extraordinary wave, whilst the latter would give rise to an ordinary wave.

Since the wave-surface of the ordinary wave is a sphere, the directions of the ordinary wave and the ordinary ray coincide; but since the wave-surface of the extraordinary wave is an ellipsoid of revolution, the directions of the extraordinary wave and the extraordinary ray do not coincide within the crystal, unless the direction of propagation is parallel or perpendicular to the axis. The question whether the vibrations of the extraordinary wave are perpendicular to the ray or the wave-normal is one, which cannot be discussed without the aid of theoretical considerations, but it may be stated that according to Fresnel's theory, the direction of vibration in the extraordinary wave is parallel to EY, that is, perpendicular to the wave-normal.

95. We have thus far given a description of the principal phenomena connected with uniaxal crystals, and of the theoretical explanation by which it is proposed to account for them, and in the next chapter we shall show how these phenomena may be explained by means of a dynamical theory. There are however certain other experimental facts which demand attention.

In all uniaxal crystals, the radius of the spherical sheet of the wave-surface is equal to the semi-polar axis of the ellipsoidal sheet; but in Iceland spar, the extraordinary index of refraction is less than the ordinary index, and therefore the ellipsoidal sheet of the wave-surface is a planetary ellipsoid. Such crystals are called *negative* crystals. There are however certain other crystals in which the ellipsoidal sheet is an ovary ellipsoid; and crystals of this kind are called *positive* crystals. It therefore follows, that for negative crystals the ellipsoidal sheet of the wave-surface lies outside the spherical sheet, whilst the converse is the case for positive crystals.

The following is a list of some of the principal uniaxal crystals.

Positive.	*Negative.*
Ice.	Beryl.
Lead hyposulphate.	Cinnabar.
Magnesium hydrate.	Emerald.
Quartz.	Iceland spar.
The red silver ores.	Ruby.
	Sapphire.
	Tourmaline.

The principal indices of refraction for Iceland spar and quartz have been determined by Rudberg, for the principal lines of the spectrum, and are as follows.

	Iceland spar		Quartz	
	μ_o	μ_e	μ_o	μ_e
B	1·65308	1·43891	1·54090	1·54990
C	1·65452	1·48455	1·54181	1·55085
D	1·65850	1·48635	1·54418	1·55328
E	1·66360	1·48868	1·54711	1·55631
F	1·66802	1·49075	1·54965	1·55894
G	1·67617	1·49453	1·55425	1·56365
H	1·68330	1·49780	1.55817	1·56772

Biaxal Crystals.

96. The investigations of Brewster and Biot showed the existence of a certain class of doubly refracting crystals, in which neither ray is refracted according to the ordinary law. Such crystals have two optic axes, and are therefore called biaxal crystals.

The form of the wave-surface for biaxal crystals was discovered by Fresnel, and is known by his name; its equation is

$$\frac{a^2x^2}{r^2-a^2}+\frac{b^2y^2}{r^2-b^2}+\frac{c^2z^2}{r^2-c^2}=0,$$

where $r^2=x^2+y^2+z^2$. The quantities a, b, c are called the principal wave velocities in the crystal. This surface will be discussed in the next chapter, but it is easy to see that if any two of the three constants a, b, c are equal, the surface splits up into a sphere and an ellipsoid of revolution.

The following is a list of some of the principal biaxal crystals.

Aragonite.
Borax.
Cerceosite (Lead carbonate).
Mica.
Nitre.
Selenite.
Sulphur.
Topaz.

97. The next table gives the values found by Rudberg for the three principal indices of refraction, of aragonite and topaz, for the principal rays of the spectrum, where μ_a denotes the ratio of the velocity of light in air, to that of the principal wave velocity a.

Rays	Aragonite			Topaz		
	μ_a	μ_b	μ_c	μ_a	μ_b	μ_c
B	1·52749	1·67631	1·68061	1·60840	1·61049	1·61791
C	1·52820	1·67779	1·68203	1·60935	1·61144	1·61880
D	1·53013	1·68157	1·68589	1·61161	1·61375	1·62109
E	1·53264	1·68634	1·69084	1·61452	1·61668	1·62408
F	1·53479	1·69053	1·69515	1·61701	1·61914	1·62652
G	1·53882	1·69836	1·70318	1·62154	1·62365	1·63123
H	1·54226	1·70509	1·71011	1·62539	1·62745	1·63506

It will be hereafter shown, that the angle between the optic axes depends upon the values of the three principal indices of refraction; and since these are slightly different for different colours, the positions of the optic axes for different colours will not coincide. This is called dispersion of the optic axes.

Quartz.

98. When plane polarized light is incident normally on a plate of Iceland spar, which is cut perpendicularly to the axis, it is found that the emergent light is polarized in the same plane as the incident light. There are however certain uniaxal crystals, of which quartz is the most conspicuous example, which possess the power of rotating the plane of polarization; that is to say, the plane of polarization of the emergent light is inclined at a certain angle to that of the incident light, which is found by experiment to be proportional to the thickness of the plate. The construction of Huygens and the theory of Fresnel do not apply to such crystals. Crystals of this class require a special theory of their own, which will be considered in the chapter on rotatory polarization.

99. Most isotropic transparent media, when subjected to stress, exhibit double refraction. For example, compressed glass acts like a negative uniaxal crystal, whose axis is parallel to the direction of compression; whilst stretched glass acts like a positive uniaxal crystal, whose axis is parallel to the axis of extension[1]. There are also certain crystals, in which the relative position of the optic axes for different colours varies with the temperature[2].

[1] Brewster, *Phil. Trans.*, 1815, p. 60.
[2] Ibid, *Phil. Trans.*, 1815, p. 1; *Phil. Mag.* (3) vol. I. p. 417.

CHAPTER VII.

FRESNEL'S THEORY OF DOUBLE REFRACTION.

100. WHEN the disturbance which produces light is excited at any point of an isotropic medium, a spherical wave is propagated from the centre of disturbance with constant velocity; but we have pointed out in the preceding chapter, that when the disturbance is excited in a doubly refracting medium, two waves are propagated with different velocities, and that when the medium is a biaxal crystal, the velocity in any given direction is a function of the inclination of this direction to the optic axes of the crystal.

The laws regulating the propagation of light in crystals, were first investigated mathematically by Fresnel, who showed that the wave-surface in biaxal crystals, is a certain quartic surface, which reduces to a sphere and an ellipsoid of revolution in the case of uniaxal crystals. The theory by means of which Fresnel arrived at this result, cannot be considered to be a strict dynamical theory; but on account of its historical interest, and also owing to the fact that experiment has proved that Fresnel's wave-surface is a very close approximation to the true form of the wave-surface in biaxal crystals, we shall proceed to explain its leading features, and afterwards discuss the geometry of this surface.

101. The theory of Fresnel depends upon the following four hypotheses, which are thus summarized by Verdet[1].

(i) *The vibrations of polarized light are perpendicular to the plane of polarization.*

(ii) *The elastic forces which are produced by the propagation of a train of plane waves, whose vibrations are transversal and*

[1] *Leçons d'Optique Physique.* Vol. I. p. 465.

rectilinear, are equal to the product of the elastic forces produced by the displacement of a single molecule of that wave, into a constant factor, which is independent of the direction of the wave.

(iii) *When a wave is propagated in a homogeneous medium, the component of the elastic forces parallel to the wave-front, is alone operative.*

(iv) *The velocity of propagation of a plane-wave, which is propagated in a homogeneous medium without change of type, is proportional to the square root of the effective component of the elastic forces developed by the vibrations of that wave.*

102. We have stated in the preceding chapter, that according to the generally received opinion, the vibrations of the ordinary wave in a uniaxal crystal are *perpendicular* to the plane containing the direction of propagation and the optic axis, whilst the vibrations of the extraordinary wave are executed *in* the corresponding plane. Up to the present time no experiments have been described which prove that this is the case, and consequently for all we know to the contrary, the vibrations of the ordinary wave might take place in the principal plane, whilst those of the extraordinary wave might be perpendicular to that plane. We shall hereafter show, that there are strong grounds for supposing, that the vibrations of polarized light are perpendicular to the plane of polarization; but for the present Fresnel's first hypothesis must be regarded as an assumption.

103. The second and third hypotheses require careful consideration, and it will be convenient to discuss them together.

Since the motions of the ether which constitute light are of a vibratory character, it follows that the ether when undisturbed, must be in stable equilibrium. Hence if $F(x, y, z) = V$ be its potential energy at any point x, y, z; and if a particle situated at this point be displaced to the point $x+u$, $y+v$, $z+w$, it follows that $dV/dx = dV/dy = dV/dz = 0$; and therefore expanding by Taylor's theorem,

$$V = V_0 + \tfrac{1}{2}(Au^2 + Bv^2 + Cw^2 + 2A'vw + 2B'wu + 2C'uv),$$

where V_0 is the constant potential energy when in equilibrium, and $A, B\ldots$ are positive constants. By properly choosing the axes, the products may be made to disappear; whence omitting the

constant term V_0, which contributes nothing to the forces, the value of V may be written

$$V = \tfrac{1}{2}(a^2u^2 + b^2v^2 + c^2w^2) \quad\text{..................(1)},$$

and therefore the forces of restitution are

$$X = a^2u, \quad Y = b^2v, \quad Z = c^2w \quad\text{..............(2)}.$$

Hence if we construct the ellipsoid

$$a^2x^2 + b^2y^2 + c^2z^2 = 1 \quad\text{....................(3)}$$

whose centre is O, and if we draw a radius OP parallel to the direction of displacement and meeting the ellipsoid in P, and OY be the perpendicular from O on to the tangent plane at P, then OY will be the direction of the resultant force.

The ellipsoid (3) is called the *ellipsoid of elasticity;* and it follows from the preceding construction, that the resultant force will not be in the direction of displacement, unless the displacement is parallel to one of the principal axes of this ellipsoid.

If l, m, n be the direction cosines of the normal to the wave-front; λ, μ, ν those of the direction of displacement, it follows that the resultant force of restitution will not be in the plane of the wave-front. This force may however be resolved into two components, one of which is in the plane of the wave, and the other is perpendicular to it. The latter component according to the third hypothesis will not give rise to vibrations which produce light, and therefore need not be considered. The former component will give rise to vibrations which produce light; but it will not coincide with the direction of displacement, unless the latter coincides with that of one or other of the principal axes of the section of the ellipsoid of elasticity by the plane $lx + my + nz = 0$. For the direction cosines of the force are proportional to $a^2\lambda, b^2\mu, c^2\nu$; and the condition that this line, the direction of the displacement, and the normal to the wave-front should lie in the same plane, is

$$\begin{vmatrix} l, & m, & n \\ \lambda, & \mu, & \nu \\ a^2\lambda, & b^2\mu, & c^2\nu \end{vmatrix} = 0,$$

or

$$\frac{l}{\lambda}(b^2 - c^2) + \frac{m}{\mu}(c^2 - a^2) + \frac{n}{\nu}(a^2 - b^2) = 0 \quad\text{........(4)}$$

which is the condition, that the line λ, μ, ν should be a principal axis of the section of the ellipsoid of elasticity by the wave-front.

Let us now suppose, that plane waves of polarized light are incident normally upon a crystalline plate, the direction cosines of whose face, with respect to the principal axes, are l, m, n. Let OA, OB be the directions of the axes of the section of the ellipsoid of elasticity made by the surface of the plate, and OP the direction of vibration of the incident light. If the second medium were isotropic instead of crystalline, a single refracted wave would be propagated, consisting of light polarized in a plane perpendicular to OP and the surface of the plate; but if the second medium is a crystal, a single wave whose vibrations are parallel to OP is incapable of being propagated, and it is necessary to suppose that the incident vibrations are resolved into two sets of vibrations, which are respectively parallel to OA, OB. These two sets of vibrations are propagated through the crystal with different velocities (unless the normal to the surface of the plate is parallel to one of the optic axes), and thus give rise to two waves of polarized light, whose planes of polarization are at right angles to one another.

104. If q be the displacement of a particle of ether in either of the waves, the equation of motion of that particle will be

$$\frac{d^2q}{dt^2} = -(a^2\lambda^2 + b^2\mu^2 + c^2\nu^2)\,q\,;$$

and therefore if τ be the time of oscillation,

$$2\pi/\tau = (a^2\lambda^2 + b^2\mu^2 + c^2\nu^2)^{\frac{1}{2}} = 2\pi v/\lambda'\,;$$

where v is the velocity of propagation, and λ' is the wave length. Hence if we write $2\pi a/\lambda'$ &c. for a, b, c, where a, b, c now denote the three principal wave velocities, we obtain

$$v^2 = a^2\lambda^2 + b^2\mu^2 + c^2\nu^2 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5).$$

From (5) it appears, that the force of restitution $a^2\lambda, b^2\mu, c^2\nu$ corresponding to a displacement unity, is equal to a force v^2 along the direction of displacement, together with some force P along l, m, n, the normal and the wave-front; whence resolving parallel to the axes, we obtain

$$lP = (a^2 - v^2)\,\lambda, \quad mP = (b^2 - v^2)\,\mu, \quad nP = (c^2 - v^2)\,\nu\,;$$

accordingly since $\qquad l\lambda + m\mu + n\nu = 0,$

it follows that
$$\frac{l^2}{v^2 - a^2} + \frac{m^2}{v^2 - b^2} + \frac{n^2}{v^2 - c^2} = 0 \quad\ldots\ldots\ldots\ldots(6),$$

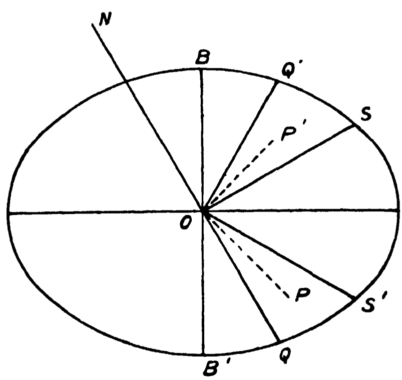
which determines the velocities of propagation of the two waves, whose direction cosines are l, m, n.

105. Before proceeding to discuss Fresnel's wave-surface, it will be convenient to consider some preliminary propositions.

We have shown that when polarized light is incident normally upon a crystal, the incident vibration must be conceived to be resolved into two components, which are parallel to the principal axes of that section of the ellipsoid of elasticity, which is parallel to the surface of the crystal; and that these two sets of vibrations give rise to two waves within the crystal. Now if the surface of the crystal is parallel to either of the circular sections of the ellipsoid of elasticity, every direction will be a principal axis, and therefore the component force parallel to the wave-front will be in the direction of displacement; hence only one wave will be propagated through the crystal. These two directions are the *optic axes* of the crystal, and therefore the optic axes are perpendicular to the two planes of circular section of the ellipsoid of elasticity.

106. We can now prove the following propositions:

The planes of polarization of the two waves corresponding to the same wave-front, bisect the angles between the two planes passing through the normal to the wave-front and the optic axes.



Let BAB' be the section of the ellipsoid of elasticity by the wave-front, ON the wave normal, and OS the intersection of one of the planes of circular section with the wave-front. The optic axis corresponding to the circular section through OS is perpendicular to OS, and therefore the plane through it and ON cuts the plane BAB' in a line OQ, which is perpendicular to OS. Similarly, if OS' be the intersection of the other plane of circular section with BAB', and OQ' be the projection of the other optic axis, OQ'

is perpendicular to OS'. Since $OS = OS'$, the angle $SOA = S'OA$, and therefore the angle $QOA = Q'OA$; hence the planes of polarization AON and BON bisect the angles between the planes QON and $Q'ON$, which are the planes containing the normal to the wave-front and the optic axes.

107. *The difference between the squares of the velocities corresponding to the same wave-front, is proportional to the product of the sines of the angles, which the normal to the wave-front makes with the optic axes.*

Let OP, OP' be the optic axes; θ, θ' the angles which they make with ON; also let λ, μ, ν be the direction cosines of OA.

The equation of the two planes of circular section are

$$x(a^2-b^2)^{\frac{1}{2}} \pm z(b^2-c^2)^{\frac{1}{2}} = 0,$$

and therefore $$\cos AP = \frac{\lambda(a^2-b^2)^{\frac{1}{2}} + \nu(b^2-c^2)^{\frac{1}{2}}}{(a^2-c^2)^{\frac{1}{2}}},$$

and $$\cos AP' = \frac{\lambda(a^2-b^2)^{\frac{1}{2}} - \nu(b^2-c^2)^{\frac{1}{2}}}{(a^2-c^2)^{\frac{1}{2}}}.$$

Since the optic axis OP lies in the plane QON, we obtain

$$\cos AP = \cos AQ \sin\theta,$$

whence $$\cos AQ \sin\theta = \frac{\lambda(a^2-b^2)^{\frac{1}{2}} + \nu(b^2-c^2)^{\frac{1}{2}}}{(a^2-c^2)^{\frac{1}{2}}}.$$

Similarly, since $AQ = AQ'$

$$\cos AQ \sin\theta' = \frac{\lambda(a^2-b^2)^{\frac{1}{2}} - \nu(b^2-c^2)^{\frac{1}{2}}}{(a^2-c^2)^{\frac{1}{2}}},$$

and therefore

$$(a^2-c^2)\cos^2 AQ \sin\theta\sin\theta' = \lambda^2(a^2-b^2) - \nu^2(b^2-c^2),$$
$$= v^2 - b^2.$$

Similarly if v' be the velocity of the other wave,

$$-(a^2-c^2)\sin^2 AQ \sin\theta\sin\theta' = v'^2 - b^2,$$

whence $$v^2 - v'^2 = (a^2-c^2)\sin\theta\sin\theta' \dots\dots\dots\dots\dots\dots (7).$$

108. Another somewhat similar formula may be obtained as follows.

We have $$\cos\theta = \cos PON = \frac{l(a^2-b^2)^{\frac{1}{2}} + n(b^2-c^2)^{\frac{1}{2}}}{(a^2-c^2)^{\frac{1}{2}}},$$

$$\cos\theta' = \cos P'ON = \frac{l(a^2-b^2)^{\frac{1}{2}} - n(b^2-c^2)^{\frac{1}{2}}}{(a^2-c^2)^{\frac{1}{2}}},$$

and therefore

$$(a^2-c^2)\cos\theta\cos\theta' = l^2(a^2-b^2) - n^2(b^2-c^2) \dots\dots (8).$$

Now v and v' are the two roots of (6), whence

$$v^2 + v'^2 = l^2(b^2+c^2) + m^2(c^2+a^2) + n^2(a^2+b^2),$$
$$= a^2 + c^2 - (a^2 - c^2)\cos\theta\cos\theta' \quad \text{..............} \quad (9),$$

by (8), and therefore from (7) and (9) we obtain

$$\left.\begin{array}{l} v^2 = a^2\sin^2\tfrac{1}{2}(\theta+\theta') + c^2\cos^2\tfrac{1}{2}(\theta+\theta') \\ v'^2 = a^2\sin^2\tfrac{1}{2}(\theta-\theta') + c^2\cos^2\tfrac{1}{2}(\theta-\theta') \end{array}\right\} \quad \text{......} \quad (10).$$

109. We are now in a position to find the equation of the wave-surface. We have already shown, that this surface is the envelop of the plane

$$lx + my + nz = v \quad \text{....................} \quad (11),$$

where l, m, n, v are subject to the condition (6), and also to the condition

$$l^2 + m^2 + n^2 = 1 \text{.......................} (12).$$

Differentiating (11), (12) and (6) with respect to l, m, n, we obtain

$$xdl + ydm + zdn - dv = 0,$$
$$ldl + mdm + ndn \qquad = 0,$$
$$\frac{ldl}{v^2-a^2} + \frac{mdm}{v^2-b^2} + \frac{ndn}{v^2-c^2} - \left\{\frac{l^2}{(v^2-a^2)^2} + \frac{m^2}{(v^2-b^2)^2} + \frac{n^2}{(v^2-c^2)^2}\right\} vdv = 0;$$

whence by indeterminate multipliers, we find

$$x + Al + \frac{Bl}{v^2-a^2} = 0 \quad \text{.................} \quad (13),$$

$$y + Am + \frac{Bm}{v^2-b^2} = 0 \quad \text{.................} \quad (14),$$

$$z + An + \frac{Bn}{v^2-c^2} = 0 \quad \text{.................} \quad (15),$$

$$1 + Bv\left\{\frac{l^2}{(v^2-a^2)^2} + \frac{m^2}{(v^2-b^2)^2} + \frac{n^2}{(v^2-c^2)^2}\right\} = 0 \text{......} (16).$$

Multiplying (13), (14) and (15) by l, m, n and adding, we obtain

$$v + A = 0 \text{.........................} (17).$$

Transposing the third terms of the same equations, squaring and adding, and remembering that $r^2 = x^2 + y^2 + z^2$, we obtain

$$r^2 + 2Av + A^2 = B^2\left\{\frac{l^2}{(v^2-a^2)^2} + \frac{m^2}{(v^2-b^2)^2} + \frac{n^2}{(v^2-c^2)^2}\right\},$$

which by (16) and (17) becomes

$$r^2 - v^2 = -B/v \qquad\qquad (18),$$

and therefore

$$\left.\begin{array}{l} x = lv\,(r^2 - a^2)/(v^2 - a^2) \\ y = mv\,(r^2 - b^2)/(v^2 - b^2) \\ z = nv\,(r^2 - c^2)/(v^2 - c^2) \end{array}\right\} \qquad\qquad (19).$$

Multiplying (13), (14), (15) by x, y, z and adding, we obtain

$$r^2 + Av + B\left(\frac{lx}{v^2 - a^2} + \frac{my}{v^2 - b^2} + \frac{nz}{v^2 - c^2}\right) = 0,$$

which by (16) and (19) becomes

$$r^2 - v^2 + \frac{B}{v}\left(\frac{x^2}{r^2 - a^2} + \frac{y^2}{r^2 - b^2} + \frac{z^2}{r^2 - c^2}\right) = 0;$$

whence by (18)

$$\frac{x^2}{r^2 - a^2} + \frac{y^2}{r^2 - b^2} + \frac{z^2}{r^2 - c^2} = 1,$$

which may be put into the more usual form

$$\frac{a^2x^2}{r^2 - a^2} + \frac{b^2y^2}{r^2 - b^2} + \frac{c^2z^2}{r^2 - c^2} = 0 \qquad\qquad (20).$$

This is the equation of Fresnel's wave-surface.

The equation may also be expressed in the form

$$r^2(a^2x^2 + b^2y^2 + c^2z^2) - a^2(b^2 + c^2)\,x^2 - b^2(c^2 + a^2)\,y^2 - c^2(a^2 + b^2)\,z^2 + a^2b^2c^2 = 0 \qquad (21),$$

which shows that the surface is a quartic surface. The preceding demonstration is due to the late Mr Archibald Smith[1].

110. We shall now consider the traces of the wave-surface on the coordinate planes.

(i) Let $x = 0$, then (21) reduces to

$$(r^2 - a^2)(b^2y^2 + c^2z^2 - b^2c^2) = 0.$$

Hence the trace of the wave-surface on the plane yz is the circle

$$y^2 + z^2 = a^2,$$

and the ellipse $\qquad y^2/c^2 + z^2/b^2 = 1.$

(ii) Let $y = 0$, then it can similarly be shown that the trace on the plane xz consists of the circle

$$x^2 + z^2 = b^2,$$

and the ellipse $\qquad x^2/c^2 + z^2/a^2 = 1.$

[1] *Trans. Camb. Phil. Soc.*, vol. VI.

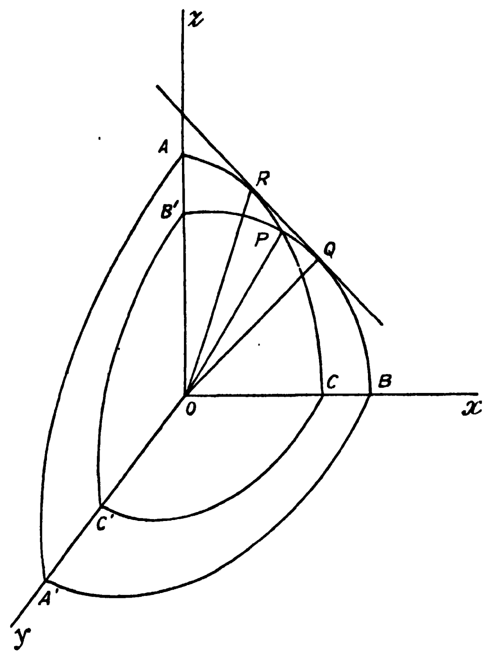
Wave surface is the locus of the extremities of the normal equal in length to the axes of the

(iii) Let $z=0$, then the trace on the plane xy, is the circle

$$x^2+y^2=c^2,$$

and the ellipse $$x^2/b^2+y^2/a^2=1.$$

The form of the surface when $a>b>c$, is shewn in the figure.



111. Since the wave-surface is symmetrical with respect to the coordinate planes, it appears that it consists of an outer and an inner sheet, which intersect at four points in the plane xz. These four points are singular points, and it will hereafter be shown, that there is a tangent and normal cone at each of them.

If QR be the common tangent in the plane xz, to the ellipse and circle, and if $xOQ=\theta$, it follows that

$$OQ^2=b^2=c^2\cos^2\theta+a^2\sin^2\theta,$$

whence $$\left.\begin{array}{l}\cos\theta=(a^2-b^2)^{\frac{1}{2}}/(a^2-c^2)^{\frac{1}{2}}\\ \sin\theta=(b^2-c^2)^{\frac{1}{2}}/(a^2-c^2)^{\frac{1}{2}}\end{array}\right\} \quad\cdots\cdots\cdots\quad (22).$$

Hence OQ is perpendicular to one of the planes of circular section of the ellipsoid of elasticity, and is therefore one of the optic axes. The other optic axis lies in the plane of xz, and makes an angle $\pi-\theta$ with the axis of x.

The line OP is called the *ray axis*, and its equation is

$$ax(b^2-c^2)^{\frac{1}{2}} = cz(a^2-b^2)^{\frac{1}{2}};$$

the ray axes are therefore perpendicular to the circular sections of the reciprocal ellipsoid

$$x^2/a^2 + y^2/b^2 + z^2/c^2 = 1.$$

112. We shall now prove, that the direction of vibration in any wave may be determined by the following simple construction.

Draw a tangent plane to the wave-surface parallel to the wave-front, touching the surface in P; then if Y be the foot of the perpendicular from the centre of the wave-surface on to this tangent plane, PY is the direction of displacement;—in other words, *the direction of vibration coincides with the projection of the ray on the wave-front.*

We have incidentally proved in § 104 that

$$(v^2-a^2)\lambda/l = (v^2-b^2)\mu/m = (v^2-c^2)\nu/n \qquad\qquad (23).$$

Combining these equations with (19) of § 109, we see that

$$(r^2-a^2)\lambda/x = (r^2-b^2)\mu/y = (r^2-c^2)\nu/z = k \text{ (say)} \qquad (24),$$

where x, y, z are the coordinates of P.

Since the equation of the tangent plane at P is

$$lx + my + nz = v,$$

it follows that if L, M, N are the direction cosines of PY, then

$$(x-lv)/L = (y-mv)/M = (z-nv)/N.$$

But by (19)

$$x - lv = \frac{x(r^2-v^2)}{r^2-a^2} = \frac{\lambda}{k}(r^2-v^2);$$

whence

$$\lambda/L = \mu/M = \nu/N.$$

113. *The ray and the direction of the resultant force are at right angles to one another.*

For the direction cosines of the ray are proportional to x, y, z; and those of the resultant force to $a^2\lambda$, $b^2\mu$, $c^2\nu$; and

$$a^2\lambda x + b^2\mu y + c^2\nu z = k\left(\frac{a^2x^2}{r^2-a^2} + \frac{b^2y^2}{r^2-b^2} + \frac{c^2z^2}{r^2-c^2}\right) = 0.$$

114. *The tangent planes to the wave-surface at the extremities of the optic axes touch the wave-surface along a circle.*

At the extremity of the optic axis OQ, $v=b$, $m=0$; and therefore by (19)

$$x = lb\,(r^2 - a^2)/(b^2 - a^2),$$
$$z = nb\,(r^2 - c^2)/(b^2 - c^2).$$

The values of l and n are given by (22), whence by substitution these equations become

$$b\,(r^2 - a^2) + x\,(a^2 - b^2)^{\frac{1}{2}}\,(a^2 - c^2)^{\frac{1}{2}} = 0,$$
$$b\,(r^2 - c^2) - z\,(a^2 - c^2)^{\frac{1}{2}}\,(b^2 - c^2)^{\frac{1}{2}} = 0.$$

These equations are satisfied by the coordinates of the points of contact of the tangent plane at the extremity of the optic axis with the wave surface, and since they represent two spheres, it follows that this tangent plane touches the wave along a circle.

The diameter of this circle is equal to QR (see fig. § 110). To find its value, let OD be that diameter of the elliptic section OCA, which is conjugate to OR. Then

$$OD\,.\,OQ = ac,$$

or

$$OD = ac/b.$$

Also

$$OR^2 + OD^2 = a^2 + c^2,$$

whence

$$\begin{aligned} QR^2 &= OR^2 - OQ^2 \\ &= a^2 + c^2 - a^2c^2/b^2 - b^2, \\ &= (a^2 - b^2)\,(b^2 - c^2)/b^2. \end{aligned}$$

115. *To find the equations of the tangent and normal cones at the singular points.*

The coordinates of P (see fig. § 110) are,

$$x = c\,(a^2 - b^2)^{\frac{1}{2}}/(a^2 - c^2)^{\frac{1}{2}}, \quad z = a\,(b^2 - c^2)^{\frac{1}{2}}/(a^2 - c^2)^{\frac{1}{2}}.$$

Substituting in (19), we obtain

$$v^2 + lv\,(a^2 - b^2)^{\frac{1}{2}}\,(a^2 - c^2)^{\frac{1}{2}}/c - a^2 = 0,$$
$$v^2 - nv\,(b^2 - c^2)^{\frac{1}{2}}\,(a^2 - c^2)^{\frac{1}{2}}/a - c^2 = 0.$$

Now l, m, n are the direction cosines of any normal through P; whence eliminating v, we obtain

$$l^2\,(a^2 - b^2) + n^2\,(b^2 - c^2) + ln\,(a^2 + c^2)\,(a^2 - b^2)^{\frac{1}{2}}\,(b^2 - c^2)^{\frac{1}{2}}/ac = a^2 - c^2,$$

whence the equation of the normal cone, referred to P as origin is

$$x^2\,(b^2 - c^2) + y^2\,(a^2 - c^2) + z^2\,(a^2 - b^2) = xz\,(a^2 + c^2)\,(a^2 - b^2)^{\frac{1}{2}}\,(b^2 - c^2)^{\frac{1}{2}}/ac \quad \text{.........} (25).$$

Let λ, μ, ν be the direction cosines of any generator of the tangent cone; then since this generator is parallel to the normal at some point of the normal cone, it follows that if $F(x, y, z)$ be the equation of the normal cone,

$$\frac{\lambda}{dF/dx} = \frac{\mu}{dF/dy} = \frac{\nu}{dF/dz},$$

and therefore since l, m, n are proportional to x, y, z in (25), we obtain

$$\frac{\lambda}{2l(b^2-c^2) - n(a^2+c^2)(a^2-b^2)^{\frac{1}{2}}(b^2-c^2)^{\frac{1}{2}}/ac} = \frac{\mu}{2m(a^2-c^2)}$$

$$= \frac{\nu}{2n(a^2-b^2) - l(a^2+c^2)(a^2-b^2)^{\frac{1}{2}}(b^2-c^2)^{\frac{1}{2}}/ac},$$

and therefore

$$-\frac{\mu}{2m} = \frac{2\lambda a^2c^2(a^2-b^2)^{\frac{1}{2}} + \nu ac(a^2+c^2)(b^2-c^2)^{\frac{1}{2}}}{l(a^2-c^2)(b^2-c^2)(a^2-b^2)^{\frac{1}{2}}}$$

$$= \frac{2\nu a^2c^2(b^2-c^2)^{\frac{1}{2}} + \lambda ac(a^2+c^2)(a^2-b^2)^{\frac{1}{2}}}{n(a^2-c^2)(a^2-b^2)(b^2-c^2)^{\frac{1}{2}}}.$$

But $$l\lambda + m\mu + n\nu = 0,$$

whence

$$\frac{\lambda^2}{b^2-c^2} - \frac{(a^2-c^2)\mu^2}{4a^2c^2} + \frac{\nu^2}{a^2-b^2} + \frac{(a^2+c^2)\lambda\nu}{ac(a^2-b^2)^{\frac{1}{2}}(b^2-c^2)^{\frac{1}{2}}} = 0,$$

and therefore the equation of the tangent cone is

$$\frac{x^2}{b^2-c^2} - \frac{(a^2-c^2)y^2}{4a^2c^2} + \frac{z^2}{a^2-b^2} + \frac{(a^2+c^2)xz}{ac(a^2-b^2)^{\frac{1}{2}}(b^2-c^2)^{\frac{1}{2}}} = 0 \quad \ldots (26).$$

116. There is a third cone which is also of importance, viz. the cone whose vertex is the origin, and whose generators pass through the circle of contact of the tangent plane at the extremity of the optic axes.

We have shown in § 114, that the circle of contact is the curve of intersection of the two spheres

$$b(r^2-a^2) + x(a^2-b^2)^{\frac{1}{2}}(a^2-c^2)^{\frac{1}{2}} = 0 \quad \ldots\ldots\ldots\ldots (27),$$

$$b(r^2-c^2) - z(a^2-c^2)^{\frac{1}{2}}(b^2-c^2)^{\frac{1}{2}} = 0 \quad \ldots\ldots\ldots\ldots (28).$$

Hence if λ, μ, ν be any generator of the required cone,

$$\frac{r^2-a^2}{r^2-c^2} = -\frac{\lambda(a^2-b^2)^{\frac{1}{2}}}{\nu(b^2-c^2)^{\frac{1}{2}}};$$

therefore $$r^2 = \frac{\nu a^2(b^2-c^2)^{\frac{1}{2}} + \lambda c^2(a^2-b^2)^{\frac{1}{2}}}{\nu(b^2-c^2)^{\frac{1}{2}} + \lambda(a^2-b^2)^{\frac{1}{2}}}.$$

Also
$$r = \frac{b\,(a^2-c^2)^{\frac{1}{2}}}{\nu\,(b^2-c^2)^{\frac{1}{2}} + \lambda\,(a^2-b^2)^{\frac{1}{2}}},$$
whence eliminating r, we obtain
$$c^2(a^2-b^2)\,\lambda^2 + a^2(b^2-c^2)\,\nu^2 + (a^2+c^2)(a^2-b^2)^{\frac{1}{2}}(b^2-c^2)^{\frac{1}{2}}\lambda\nu = (a^2-c^2)\,b^2$$
and therefore the equation of the cone is
$$a^2(b^2-c^2)\,x^2 + b^2(a^2-c^2)\,y^2 + c^2(a^2-b^2)\,z^2 - (a^2+c^2)(a^2-b^2)^{\frac{1}{2}}(b^2-c^2)^{\frac{1}{2}}\,xz = 0 \quad\ldots\ldots\ldots(29).$$

Uniaxal Crystals.

117. If in equation (21) we put $b = c$, it becomes
$$(r^2-c^2)\,\{a^2x^2 + c^2(y^2+z^2) - a^2c^2\} = 0,$$
which is the form of the wave-surface for a uniaxal crystal. Hence the wave-surface consists of the sphere
$$x^2 + y^2 + z^2 = c^2,$$
and the ellipsoid $\qquad a^2x^2 + c^2(y^2+z^2) = a^2c^2,$

the axis of x being the axis of revolution.

Also from (22) we see that when $b = c$, $\theta = 0$; whence the two optic axes coincide with the axis of x, which is therefore the axis of the crystal.

The ellipsoid is ovary or planetary, according as $c >$ or $< a$. In the former case the crystal is positive, and in the latter case negative.

If a pencil of light be incident upon a uniaxal crystal, the ray corresponding to the spherical sheet of the wave-surface, will coincide with the wave normal, and refraction will take place according to the ordinary law discovered by Snell. Also if λ, μ, ν be the direction cosines of the direction of vibration, we obtain from (5),
$$\begin{aligned} c^2 &= a^2\lambda^2 + c^2(\mu^2+\nu^2) \\ &= a^2\lambda^2 + c^2(1-\lambda^2); \end{aligned}$$
whence $\lambda = 0$, which shows that the direction of vibration is *perpendicular* to the plane containing the normal to the wave-front and the optic axis.

The extraordinary ray is in the direction of the radius vector of the ellipsoidal sheet of the wave-surface, drawn to the point of contact of the tangent plane, which is perpendicular to the wave

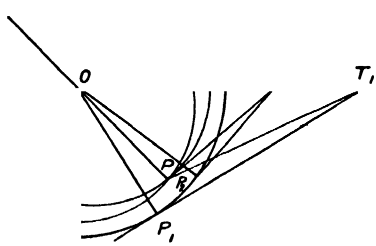
normal; and by § 112 the direction of vibration is the projection of the ray on the wave-front. Hence the direction of vibration in the extraordinary wave, lies *in* the plane containing the optic axis and the extraordinary wave-normal, and is perpendicular to the latter.

We have thus established the laws of the propagation of light in uniaxal crystals, which were discovered experimentally by Huygens.

Conical Refraction.

118. The existence of the tangent cone at the extremity of the ray axis was first demonstrated by Sir W. Hamilton, and this led to the discovery of two remarkable phenomena, known as *external and internal conical refraction.*

119. In order to explain external conical refraction, let us suppose that a small pencil of light is incident upon a plate of biaxal crystal, cut perpendicularly to the line bisecting the acute angle between the optic axes; and let the angle of incidence be such, that the direction of the refracted ray within the crystal coincides with the ray axis.



Let IO be the ray axis within the crystal, I being the point of incidence, and O the point of exit. At O draw the wave-surface for the crystal, and also the equivalent sphere in air. Produce IO to meet the crystalline wave-surface in P; then OP will be the ray axis. To obtain the directions of the refracted rays, draw tangent planes at P. These tangent planes will meet the face of the crystal in a series of straight lines $T_1, T_2 \ldots$; through each of these straight lines $T_1, T_2 \ldots$ draw a tangent plane to the sphere,

and draw OP_1, OP_2... joining the points of contact with O. The points of contact of the infinite number of tangent planes to the sphere will lie on a certain spherical curve, and therefore the refracted rays on emerging from the crystal, will form a conical pencil whose vertex is O, and whose generators are the lines OP_1, OP_2....

120. In order to explain internal conical refraction, we must suppose that the angle of incidence is such, that the direction of the refracted wave coincides with the optic axis. Since the tangent plane at the extremity of the optic axis touches the wave-surface along a circle, the refracted pencil within the crystal, will consist of a cone of rays, whose vertex is the point of incidence, and all of whose generators pass through the above-mentioned circle. The equation of this cone is given by (29). On emerging from the crystal, each emergent ray will be parallel to the incident ray, and will form an emergent cylinder of rays.

121. The phenomena of external and internal conical refraction had never been observed nor even suspected, previously to the theoretical investigations of Sir W. Hamilton on the geometry of the singular points of the wave-surface; and at his suggestion, Dr Humphrey Lloyd[1] examined the subject experimentally, and found that both kinds of conical refraction actually existed.

122. The investigations of Sir W. Hamilton, coupled with the experiments of Dr Lloyd, are undoubtedly a striking confirmation of the accuracy of Fresnel's wave-surface; but it has been subsequently pointed out by Sir G. Stokes[2], that almost any theory which could be constructed, would lead to a wave surface having conical points, and would therefore account for the phenomenon of conical refraction. Also a series of very elaborate experiments by Glazebrook[3] upon uniaxal and biaxal crystals, have shown that Fresnel's wave-surface does not quite accurately represent the true form of the wave-surface in such crystals, but is only a very close approximation.

123. The dynamical objections to Fresnel's theory may be classed under three heads.

[1] *Trans. Roy. Ir. Acad.* vol. XVII. p. 145.
[2] *Brit. Assoc. Rep.* 1862.
[3] *Phil. Trans.* 1879, p. 287; 1880, p. 421.

(i) It is assumed that the potential energy of an elastic medium, which is displaced from its position of equilibrium, is a quadratic function of the component displacements; whereas it will be shown in a subsequent chapter, that the potential energy of an elastic medium, which is symmetrical with respect to three rectangular planes, is a certain quadratic function which involves the space variations of the displacements, and not the displacements themselves.

(ii) The component of the force of restitution, perpendicular to the direction of propagation of the wave, is altogether neglected. And although attempts may be made to justify this by arguing, that the effect of this force, whatever it may be, cannot give rise to vibrations which affect the eye, yet the argument is fallacious; inasmuch as if such forces existed, they would produce waves of longitudinal vibrations, which would give rise to transversal vibrations, when light passes from a crystalline medium into another medium, and thus the sensation of light would be produced by something which is not light.

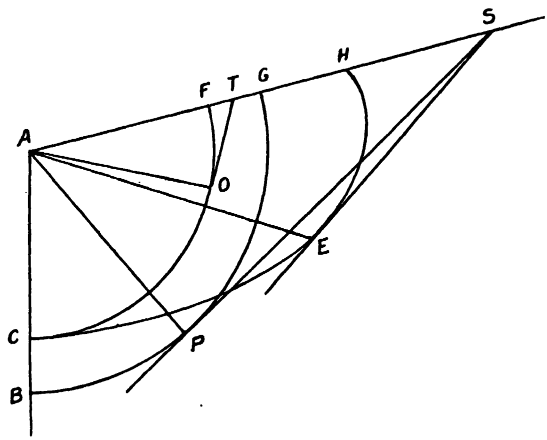
(iii) It is not a legitimate way of dealing with the motion of an elastic medium, to treat a wave as if it were composed of a number of distinct particles, each of which is acted upon by a force depending on its displacement. The rigorous theory of æolotropic elastic media is due to Green, and will be considered in a subsequent chapter; but although this theory is rigorous as far as its dynamics are concerned, it does not offer a satisfactory explanation of double refraction.

On the Methods of producing Polarized Light.

124. When light falls upon a plate of Iceland spar, it is divided into two rays within the crystal, which are polarized in perpendicular planes, and on emerging from the plate, two streams of plane polarized light are obtained, which are parallel to the incident rays; but unless the thickness of the plate is considerable, these two streams overlap. Since the velocities of the two streams within the crystal are unequal, their phases on emergence are different, and consequently the emergent beam is elliptically polarized, unless the difference of phase amounts to a quarter of a wave-length, in which case it is circularly polarized.

125. A very convenient method of producing plane polarized light consists in passing common light through a Nicol's prism, so called after the name of its inventor, the construction of which we shall proceed to explain.

There is a transparent substance called Canada balsam, whose index of refraction is intermediate between the ordinary and extraordinary indices of refraction of Iceland spar. If therefore two rhombs of Iceland spar are cemented together with this substance, it is possible for the ordinary ray to be totally reflected at the surface of the balsam, so that the extraordinary ray is alone transmitted.



Let AC be the optic axis, $ACOF$, $ACEH$ the spherical and spheroidal sheets of the wave-surface; and let the plane of the paper be the plane of incidence, which is supposed to contain the optic axis. Let AO, AE be the ordinary and extraordinary rays, corresponding to a ray incident at A.

Let ABG be the wave-surface of the balsam; then since the index of refraction of the latter is intermediate between the ordinary and extraordinary indices of the spar, ABG will be a sphere, whose radius is intermediate between the polar and equatorial axes of the spheroid.

In order to obtain the directions within the balsam of the ray corresponding to the ordinary ray, draw a tangent at O meeting the face of the spar in T, and from T draw a tangent to ABG, and join the point of contact with A; if however T lies between F and G, it will be impossible to draw this tangent, and the ordinary ray will be totally reflected.

whence to a sufficient approximation,

$$\sin\omega \sin i = y/d, \quad \cos\omega \sin i = x/d,$$

and therefore the equation of the isochromatic curves is

$$ay^2 - bx^2 = \frac{2d^2v^2}{b}\left\{\frac{ab(2n+1)\lambda}{2Tv(a-b)} - 1\right\} \quad\text{............(10)},$$

and are therefore hyperbolas, whose asymptotes are the straight lines

$$y = \pm (b/a)^{\frac{1}{2}}x.$$

Since the right-hand side of (10) does not in general vanish, the asymptotes do not usually form part of the system of isochromatic curves.

If $\alpha = \frac{1}{4}\pi$, $\beta = \frac{1}{2}\pi$, the intensity is equal to $\sin^2\pi(O-E)/\lambda$; whence the dark rings are given by the equation $O - E = n\lambda$, and are therefore complementary to the bright rings in the preceding case.

136. When the axis is neither parallel nor perpendicular to the surface of the plate, the calculation becomes more complicated. The isochromatic curves are of the fourth degree, which approximate to circles when the axis is nearly perpendicular to the plate, and to hyperbolas when the axis is nearly parallel to the plate. For the mathematical investigation, the reader is referred to Verdet's *Leçons d'Optique Physique*, vol. II. p. 161.

Two Plates Superposed.

137. We shall now suppose, that light passes through two plates cut parallel to the axis, which are of the same thickness and are cut from the same piece of crystal, and that their principal planes are at right angles.

In the figure to § 130, let OA, OB be the principal planes of the first and second plates respectively; then we have shown in § 130, that on emergence from the first plate, the vibrations may be represented by

$$\cos\alpha \sin 2\pi(t/\tau - E/\lambda)$$

along OA, and

$$\sin\alpha \sin 2\pi(t/\tau - O/\lambda)$$

along OB.

The first of these waves, which is the extraordinary wave in the first plate, becomes the ordinary wave in the second plate; whilst the second becomes the extraordinary wave. Hence if O', E' denote the retardations produced by the second plate measured by their equivalent paths in air, the vibrations on emergence will be represented by

$$\cos\alpha \sin 2\pi \{t/\tau - (O' + E)/\lambda\}$$

along OA, and

$$\sin\alpha \sin 2\pi \{t/\tau - (O + E')/\lambda\}$$

along OB.

Since the crystals are of the same thickness and of the same material, we must have $O = O'$; whence if $\phi = t/\tau - (O + E)/\lambda$, these become

$$\cos\alpha \sin 2\pi\phi, \text{ and } \sin\alpha \sin 2\pi \{\phi + (E - E')/\lambda\},$$

and therefore on emerging from the analyser, the resultant vibration is

$$\cos\alpha \cos(\alpha - \beta) \sin 2\pi\phi + \sin\alpha \sin(\alpha - \beta) \sin 2\pi \{\phi + (E - E')/\lambda\};$$

accordingly the intensity is equal to

$$I^2 = \cos^2\beta - \sin 2\alpha \sin 2(\alpha - \beta) \sin^2 \pi (E - E')/\lambda.$$

The value of the quantity $O - E'$ is obtained from (9) by putting $\frac{1}{2}\pi - \omega$ for ω, whence

$$E - E' = \tfrac{1}{2}T(a^2 - b^2) \sin^2 i (\cos^2\omega - \sin^2\omega)/av.$$

The appearance presented on a screen can be discussed in this case in the same manner as in the preceding. The most favourable cases for the production of the rings are when $\alpha = \frac{1}{4}\pi$, and $\beta = 0$ or $\frac{1}{2}\pi$. In the first case the intensity is equal to $\cos^2 \pi (E - E')/\lambda$, whence the bright rings are given by the equation

$$E - E' = n\lambda$$

or

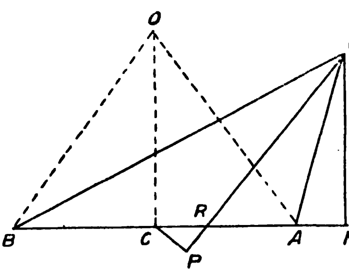
$$x^2 - y^2 = 2n\lambda a v d^2/(a^2 - b^2) T$$

where n is zero or any positive or negative integer. The isochromatic curves therefore consist of the two systems of rectangular hyperbolas, which are included in the equation $x^2 - y^2 = \pm k^2$, together with their asymptotes $y = \pm x$.

Biaxal Crystals.

138. We shall first consider the rings and brushes produced by a plate of biaxal crystal, such as nitre or aragonite, whose optic axes make a small angle with one another, and which is cut perpendicularly to the axis of least elasticity.

We shall first find the form of the brushes.



In the figure, let O be the point of incidence for any ray; let OQ be either of the wave normals within the crystal corresponding to this ray. Let the plane of the paper be any plane parallel to the face of the crystal, and let A, B and C be the points where the two optic axes and the axis of least elasticity, corresponding to O, meet the plane of the paper.

We have already shown, that the plane of polarization of this wave bisects either the internal or the external angle between the planes OQA and OQB; and since these angles, and also the angle AOB are small, the intersection of the plane of polarization with the plane of the paper, will coincide very nearly with the internal or external bisector of the angle AQB. Hence these bisectors will fix the positions of the principal planes at Q. From (1) we see that the intensity is constant when $\alpha = \frac{1}{2}n\pi$, or $\alpha - \beta = \frac{1}{2}n\pi$; hence the brushes are the loci of all points, at which the principal plane is parallel or perpendicular to the planes of polarization or analysation. Let CP be perpendicular to the plane of polarization; and let Q be chosen so that QR, the internal bisector of the angle AQB, is perpendicular to CP. Then $\alpha = PRC = ARQ$; hence if $CN = x$, $QN = y$, $AB = 2a$, $AQR = z$,

$$\frac{y}{x-a} = \tan NAQ = \tan(\alpha + \epsilon)$$

$$\frac{y}{x+a} = \tan NBQ = \tan(\alpha - \epsilon)$$

whence $$\frac{2xy}{x^2 - y^2 - a^2} = \tan 2\alpha;$$

accordingly the brushes are the rectangular hyperbolas determined by the equation

$$x^2 - y^2 - 2xy \cot 2\alpha = a^2 \text{.................} (11).$$

A similar method of proof will show, that when QP is perpendicular to the plane of analysation, the brushes are the hyperbolas

$$x^2 - y^2 - 2xy \cot 2(\alpha - \beta) = a^2 \text{..............} (12).$$

From (11) and (12) we see, that the brushes consist of two rectangular hyperbolas, each of which passes through the extremities of the optic axes.

When $\beta = \frac{1}{2}n\pi$, so that the planes of polarization and analysation are either parallel or perpendicular to one another, the two hyperbolas coincide; in the former case $\beta = 0$, the brush is light, whilst in the latter case $\beta = \frac{1}{2}\pi$, and the brush is dark.

When $\alpha = \frac{1}{2}n\pi$, and $\beta = \frac{1}{2}\pi$ so that the plane of polarization is parallel or perpendicular to the plane containing the optic axes, and the Nicols are crossed, (11) and (12) both reduce to $xy = 0$; whence the brushes consist of a dark cross.

When $\alpha = \frac{1}{4}\pi$, $\beta = \frac{1}{2}\pi$, (11) and (12) become

$$x^2 - y^2 = a^2$$

so that the brushes consist of a rectangular hyperbola, whose transverse axis is the line joining the extremities of the optic axes.

139. In order to ascertain the form of the rings, we require to evaluate the expression $(\cot r - \cot r')\sin i$. A very complete analytical investigation will be found in Verdet[1]; but the problem may be solved with sufficient approximation as follows.

By means of (4), the right-hand side of (5) may be expressed in the form

$$T\{u^{-1}(v^2 - u^2\sin^2 i)^{\frac{1}{2}} - u'^{-1}(v^2 - u'^2\sin^2 i)\}$$

which is equal to

$$T\left\{v\left(\frac{1}{u} - \frac{1}{u'}\right) - \frac{1}{2v}(u - u')\sin^2 i + \dots\dots\right\}.$$

Since i, and the difference between u and u' are small, we may neglect the terms in $\sin^2 i$, whence (5) becomes

$$O - E = Tv\left(\frac{1}{u} - \frac{1}{u'}\right) \text{....................} (13).$$

[1] *Leçons d'Optique Physique*, vol. II. p. 170; see also Bertin, *Ann. de Chim. et de Phys.* vol. LXIII. p. 57 (1861).

Since the angle between the optic axes, and also the angle of incidence are very small, the two rays, and also the two wave normals corresponding to any incident ray, may be approximately supposed to coincide with OQ; hence in the formulæ (10), of § 108,

$$2u^2 = a^2 + c^2 - (a^2 - c^2)\cos(\theta + \theta'),$$

$$2u'^2 = a^2 + c^2 - (a^2 - c^2)\cos(\theta - \theta'),$$

we may suppose that θ, θ' denote the angles QOA and QOB respectively.

Since θ, θ' are small, we have

$$u^2 = c^2 + \tfrac{1}{4}(a^2 - c^2)(\theta + \theta')^2,$$

$$u'^2 = c^2 + \tfrac{1}{4}(a^2 - c^2)(\theta - \theta')^2,$$

whence $$\frac{1}{u} - \frac{1}{u'} = -\frac{(a^2 - c^2)\,\theta\theta'}{2c^3}$$

$$= -\frac{(a^2 - c^2)\,QA\,.\,QB}{2c^3\,OA^2} \quad \text{.................}(14).$$

The isochromatic curves are determined by the condition that

$$O - E = n\lambda;$$

and consequently by (13) and (14), they consist of a family of lemniscates, whose foci are the extremities of the optic axes.

The form of the rings and brushes when $\alpha = \frac{1}{2}\pi$, $\beta = \frac{1}{2}\pi$ are shown in figure 3; and when $\alpha = \frac{1}{4}\pi$, $\beta = \frac{1}{2}\pi$ in figure 4, of the plate at the end of Chapter IX.

It may be worth while to point out, that the rings and brushes are both included in the equation

$$(x + \iota y)^2 - c^2 = c^2 \epsilon^{2(\xi + \iota\eta)},$$

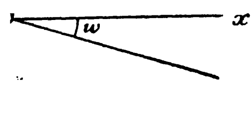
in which the curves $\xi = \text{const.}$ are lemniscates, and $\eta = \text{const.}$ are rectangular hyperbolas.

140. When the optic axes form an angle which is very nearly equal to 180°, the crystal approximates to a uniaxal crystal, which is cut parallel to the axis, and it may be anticipated that the rings are hyperbolas. This we shall show to be the case.

The velocity of propagation corresponding to the wave-front l, m, n is determined by the equation

$$\frac{l^2}{V^2 - a^2} + \frac{m^2}{V^2 - b^2} + \frac{n^2}{V^2 - c^2} = 0 \quad \text{...........}(15).$$

In the figure, let ω be the angle which the plane of incidence corresponding to any ray makes with the plane containing the



optic axes; i, r the angle of incidence and refraction. Then

$$l = \cos\omega \sin r, \quad m = \sin\omega \sin r, \quad n = \cos r;$$

also
$$V = v \sin r \operatorname{cosec} i.$$

Writing (15) in the form

$$V^4 - V^2 \{l^2(b^2+c^2) + m^2(c^2+a^2) + n^2(a^2+b^2)\} + l^2b^2c^2 + m^2c^2a^2 + n^2a^2b^2 = 0,$$

and then substituting the above values of l, m, n, V, we shall finally obtain after reduction

$$\begin{aligned} a^2b^2 \cot^4 r \sin^4 i &+ [\{a^2b^2 + c^2(b^2\cos^2\omega + a^2\sin^2\omega)\}\sin^2 i \\ &- v^2(a^2+b^2)]\cot^2 r \sin^2 i + c^2\sin^4 i\,(b^2\cos^2\omega + a^2\sin^2\omega) \\ &- v^2\sin^2 i\,(b^2\cos^2\omega + a^2\sin^2\omega + c^2) + v^4 = 0 \quad \ldots\ldots\ldots\ldots(16). \end{aligned}$$

This is a quadratic equation for determining the two values of $\cot^2 r \sin^2 i$, corresponding to a given angle of incidence i.

Let
$$h^2 = b^2\cos^2\omega + a^2\sin^2\omega.$$

Then

$$2a^2b^2\cot^2 r\sin^2 i = v^2(a^2+b^2) - (a^2b^2 + c^2h^2)\sin^2 i \pm H \quad \ldots (17),$$

where

$$\begin{aligned} H^2 &= \{v^2(a^2+b^2) - (a^2b^2+c^2h^2)\sin^2 i\}^2 - 4a^2b^2(c^2h^2\sin^4 i - v^2h^2\sin^2 i + v^4) \\ &= (a^2b^2 - c^2h^2)^2\sin^4 i - 2v^2\sin^2 i\{(a^2b^2+c^2h^2)(a^2+b^2) - 2a^2b^2(h^2+c^2)\} \\ &\qquad + v^4(a^2-b^2)^2 \\ &= \{b^2(a^2-c^2)\cos^2\omega + a^2(b^2-c^2)\sin^2\omega\}^2\sin^4 i \\ &\quad - 2(a^2-b^2)\{b^2(a^2-c^2)\cos^2\omega + a^2(b^2-c^2)\sin^2\omega\}v^2\sin^2 i + v^4(a^2-b^2)^2 \\ &\qquad + 4a^2v^2(a^2-b^2)(b^2-c^2)\sin^2\omega\sin^2 i \\ &= [v^2(a^2-b^2) - \{b^2(a^2-c^2)\cos^2\omega + a^2(b^2-c^2)\sin^2\omega\}\sin^2 i]^2 \\ &\qquad + 4a^2v^2(a^2-b^2)(b^2-c^2)\sin^2\omega\sin^2 i \quad \ldots\ldots\ldots (18). \end{aligned}$$

We have shown in § 111, that if $2A$ be the angle between the optic axes, $\cos A = (b^2 - c^2)^{\frac{1}{2}}/(a^2 - c^2)^{\frac{1}{2}}$;
hence if A be nearly equal to $90°$, $b - c$ will be very small, and we may therefore neglect the term $(b^2 - c^2)\sin^2 i$; accordingly (18) may be written $H = v^2(a^2 - b^2) - b^2(a^2 - c^2)\cos^2\omega\sin^2 i$.

Now

$$a^2b^2 + c^2h^2 = b^2(a^2 + c^2)\cos^2\omega + 2a^2c^2\sin^2\omega + a^2(b^2 - c^2)\sin^2\omega;$$

the last term may be omitted when multiplied by $\sin^2 i$, whence by (17) we obtain

$$b^2\cot^2 r_1\sin^2 i = v^2 - (b^2\cos^2\omega + c^2\sin^2\omega)\sin^2 i,$$

$$a^2b^2\cot^2 r_2\sin^2 i = v^2b^2 - c^2(b^2\cos^2\omega + a^2\sin^2\omega)\sin^2 i;$$

whence

$$(\cot r_1 - \cot r_2)\sin i = \frac{1}{vab}\left[(a - b)v^2 - \tfrac{1}{2}\left\{a(b^2\cos^2\omega + c^2\sin^2\omega) - \frac{c^2}{b}(b^2\cos^2\omega + a^2\sin^2\omega)\right\}\sin^2 i\right].$$

To find the isochromatic curves, we must write $x/d = \cos\omega\sin i$, $y/d = \sin\omega\sin i$, and we shall find that these curves are determined by the equation

$$b^2(c^2 - ab)x^2 + ac^2(a - b)y^2 = \text{const.}$$

In the preceding investigation, we have tacitly supposed that the axis of least elasticity is perpendicular to the plate, in which case $a > b > c$; accordingly $ab > c^2$, and the curves are hyperbolas. Also since the constant on the right-hand side may be either positive or negative, we see that there are two systems of hyperbolas.

The investigation would however equally apply to a crystal such as nitre or aragonite, whose optic axes make a small angle with one another, and which is cut perpendicularly to the *greatest* axis of elasticity. In this case, we must suppose that c is the greatest and a is the least principal wave velocity; whence $c^2 > ab$, but $a < b$, so that the curves are still hyperbolas.

141. When a biaxal crystal is cut perpendicularly to either of the optic axes, the isochromatic curves are, as might be expected, approximately ellipses, which are symmetrical with respect to this axis. The equation to these curves can be shown to be

$$\{b^2(a^2 - c^2) - (a^2 - b^2)\}x^2 + a^2(b^2 - c^2)y^2 = \text{const.},$$

the velocity of light in air being taken as unity (see Verdet, vol. II. p. 179).

Circularly polarized Light.

142. We have hitherto supposed that the incident light is plane polarized, and that it is analysed by an instrument which could plane polarize common light. If however the light which has passed through the polarizer, or the light which emerges from the crystal, is passed through an apparatus which could circularly polarize plane polarized light, the rings and brushes undergo certain modifications, which we shall proceed to consider.

143. Circularly polarized light may be either produced by passing plane polarized light through a Fresnel's rhomb, which is an instrument which will be explained in a subsequent chapter; or by passing it through a quarter undulation plate, which consists of a thin plate of uniaxal crystal cut parallel to the axis, of such a thickness, that it produces a difference between the retardations of the ordinary and extraordinary waves, which is equal to a quarter of a wave-length.

Let γ be the angle which the principal section of the quarter undulation plate makes with the plane of polarization of the incident light. Then since $O - E = \frac{1}{4}\lambda$, the vibrations on emergence parallel and perpendicular to the principal section of the quarter undulation plate, are

$$\sin\gamma \cos 2\pi (t/\tau - O/\lambda), \quad \text{and} \quad \cos\gamma \sin 2\pi (t/\tau - O/\lambda).$$

We therefore see that the effect of the plate is to convert plane polarized light into elliptically polarized light; if however $\gamma = \frac{1}{4}\pi$, the emergent light is circularly polarized.

144. We shall first suppose, that the quarter undulation plate (or the Fresnel's rhomb) is placed between the polarizer and the crystal, so that the light incident upon the latter is circularly polarized. The vibrations incident upon the crystal may be taken to be $\cos 2\pi t/\tau$ in the principal plane, and $\sin 2\pi t/\tau$ perpendicularly to the principal plane; hence if $\phi = 2\pi (t/\tau - O/\lambda)$, the vibration on emerging from the analyser is

$$\cos\{\phi + 2\pi (O - E)/\lambda\} \cos\alpha + \sin\phi \sin\alpha,$$

where α is the angle which the principal section of the analyser makes with the principal plane of the incident ray. Whence the intensity of the emergent light, is proportional to

$$I^2 = 1 - \sin 2\alpha \sin 2\pi (O - E)/\lambda \ldots\ldots\ldots\ldots\ldots (19).$$

From this expression we see, that I^2 can never vanish unless $\sin 2\alpha \sin 2\pi (O - E)/\lambda = 1$; hence there are no brushes.

When the crystal is a plate of Iceland spar, cut perpendicularly to the axis, $O - E$ varies as r^2, whence (19) may be written

$$I^2 = 1 - \sin 2\alpha \sin kr^2 \qquad \text{(20)},$$

where k is a constant.

If we assign any constant value to r, say $r^2 = (2n + \frac{1}{2})\pi/k$, the intensity along this circle is zero at the points $\alpha = \frac{1}{4}\pi$, or $\frac{5}{4}\pi$, and a maximum when $\alpha = \frac{3}{4}\pi$, or $\frac{7}{4}\pi$. When $r^2 = n\pi/k$, I^2 is constant, and equal to 1; and when $r^2 = (2n + \frac{3}{2})\pi/k$, the intensity is a maximum when $\alpha = \frac{1}{4}\pi$ or $\frac{5}{4}\pi$, and a minimum when $\alpha = \frac{3}{4}\pi$ or $\frac{7}{4}\pi$. Hence the general appearance of the pattern is, that the brushes are absent, whilst the rings in the first and third quadrants are pulled out, whilst those in the second and fourth are pushed in.

Any other case can be discussed in a similar manner; and the appearance is of an analogous character, when the incident light is plane polarized and circularly analysed.

145. We shall lastly consider the case in which the light is circularly polarized and circularly analysed. In order to accomplish this, we must place another quarter undulation plate between the crystal and the analyser, with its principal plane inclined at an angle of 45° to the principal plane of the latter.

Let γ be the angle between the principal section of the quarter undulation plate, and that of the crystal. Then on emerging from the plate, the vibrations parallel and perpendicular to the principal section are of the form

$$\cos\chi \sin\gamma - \sin\{\chi + 2\pi(O - E)/\lambda\}\cos\gamma,$$

and

$$\sin\chi\cos\gamma - \cos\{\chi + 2\pi(O - E)/\lambda\}\sin\gamma;$$

whence on emerging from the analyser, the vibration is

$$\sin(\chi + \gamma) - \sin\{\chi + \gamma + 2\pi(O - E)/\lambda\};$$

accordingly

$$I^2 = 4\sin^2\pi(O - E)/\lambda.$$

It therefore follows that the rings are of the same form as when the light is plane polarized and analysed, but that there are no brushes.

EXAMPLES.

1. If a horizontal ray is first polarized in a vertical plane, then passed through a plate of crystal with its axis inclined at an angle $\frac{1}{4}\pi$ to the vertical, then through a film which retards by a quarter undulation, light polarized in a vertical plane; show that the emergent light is polarized in a plane inclined to the vertical, at an angle equal to half the retardation of phase due to the plate of crystal.

2. Plane polarized light is incident normally on a plate of uniaxal crystal cut parallel to its axis, and is then passed through a parallel plate of crystal, which could circularly polarize plane polarized light. Prove that the emergent light will be plane polarized if

$$\tan \alpha = \tan \beta \sin 2\pi k/\lambda - \tan \gamma \cos 2\pi k/\lambda;$$

where α is the angle between the principal plane of the first plate and the plane of polarization of the incident light, β is the angle between the principal plane of the second plate and the plane of polarization of the emergent light, γ is the angle between the principal planes of the first and second plates, and k is the equivalent in air to the relative retardation of the ordinary and extraordinary rays caused by the first plate.

3. A small beam of circularly polarized light is incident on one of the parallel faces of a plate of uniaxal crystal, which is cut parallel to its axis, the angle of incidence being small; and the crystal is then made to revolve round a common normal to its plane faces, whilst the direction of the incident pencil remains unchanged. It is found, that when the axis of the crystal lies in the plane of incidence, the emergent light is circularly polarized in the opposite direction to the incident light; and when the axis of the crystal is at right angles to the plane of incidence, the emergent light is circularly polarized in the same direction as the incident light. Prove that if the axis of the crystal were inclined at an angle $\frac{1}{4}\pi$ to the plane of incidence, the emergent light would be polarized either in or perpendicularly to that plane.

4. If n equal and similar plates of a crystal be laid upon each other, with their principal directions arranged like steps of a uni-

form spiral staircase, and a polarized ray pass normally through them; prove that the component vibrations of the emergent ordinary and extraordinary rays are of the form

$$X \cos 2\pi t/\tau + Y \sin 2\pi t/\tau,$$

where X and Y are of the form $A \cos n\gamma + B \sin n\gamma$, where $\cos \gamma = \cos \delta \cos \alpha$; α being the angle between the principal directions of two consecutive plates, and 2δ the difference of phase between the ordinary and extraordinary rays in passing through one plate.

Determine also the condition, that a ray originally plane polarized may emerge plane polarized.

5. The extraordinary wave normal OQ in a uniaxal crystal, whose optic axis is OA, makes a constant direction with a given direction OP. Show that the mean of the displacements irrespective of sign, which are parallel to the plane POA as the position of OQ varies, will be a minimum, when OA and OP are at right angles to one another.

6. If a biaxal crystal is bounded by two parallel planes perpendicular to the axis of greatest elasticity, and if θ, ϕ be the angles of inclination to this axis of the two emergent rays, situated in the plane containing the optic axes at the point of emergence, prove that

$$b^2c^2 \operatorname{cosec}^2 \theta = a^2 (a^2 \cot^2 \phi + c^2).$$

7. A pencil passing through a feebly doubly refracting plate, is defined by two small holes through which it has to pass, the holes being situated in a line perpendicular to the plate, and on opposite sides of it; show that whatever be the law of double refraction, when the thickness of the plate and the distances of the holes vary, the angle in air between the two pencils which can pass, varies as

$$\frac{h}{h + \mu (k + k')}$$

where h is the thickness of the plate, and k, k' the distances of the holes from the surfaces respectively next them.

8. The surfaces of a plate of uniaxal crystal are nearly perdicular to the axis of the crystal; show that if polarized light be incident nearly perpendicularly to the faces, and afterwards analysed and received on a screen, the rings will be sensibly the same as

would have been formed if the surfaces had been perpendicular to the axis, but shifted in the direction of the projection of the axis through a distance proportional to $\mu\alpha$, where μ is the refractive index for the ordinary ray, and α the angle between the axis of the crystal and a normal to the surfaces of the plate.

9. Plane polarized light of amplitude c passes in succession through two plates of crystal cut parallel to the axis, and is then analysed by a Nicol's prism. The inclinations of the principal planes of the two crystals, and of the Nicol's prism to the plane of polarization of the incident light, are α, $\alpha+\beta$, and $\alpha+\beta+\gamma$; p and q are the retardations of phase due to the two crystals respectively. Prove that if AO be drawn equal to $c \cos\alpha \cos\beta \cos\gamma$,

$$AB = c \sin\alpha \sin\beta \cos\gamma, \qquad BC = c \cos\alpha \sin\beta \sin\gamma,$$

and
$$CD = c \sin\alpha \cos\beta \sin\gamma;$$

and if AB, BC, CD make with AO angles respectively equal to p, q and $p+q$; then OD will be the amplitude, and the supplement of CDO will be the retardation of phase of the emergent ray.

10. The end of a Nicol's prism, in which air is substituted for balsam, is a rhombic face inclined at an angle $\frac{1}{4}\pi$ to the axis of the crystal, and the prism is sawn so that the layer of air contains that axis. If the axes of the ellipsoid in the wave-surface corresponding to a sphere of unit radius in air be $(2{\cdot}3)^{-\frac{1}{2}}$, $(2{\cdot}6)^{-\frac{1}{2}}$, the cosines of the angles of incidence for the extinction of the ordinary and extraordinary rays are respectively equal to

$$\frac{2}{\sqrt{5}} - \frac{1}{\sqrt{2}}, \text{ and } \sqrt{\frac{46}{65}} - \frac{1}{\sqrt{2}}.$$

CHAPTER IX.

ROTATORY POLARIZATION.

146.* WHEN plane polarized light is transmitted at normal incidence through a plate of Iceland spar, which is cut perpendicularly to the axis, the plane of polarization of the emergent light coincides with that of the incident light. It was however discovered by Arago[1] in 1811, that there are certain uniaxal crystals, of which quartz is the most notable example, which possess the power of rotating the plane of polarization. It thus appears, that crystals of the class to which quartz belongs possess certain peculiarities, which distinguish them from ordinary uniaxal crystals, such as Iceland spar.

The subject of the rotation of the plane of polarization by crystals was afterwards studied experimentally by Biot[2], who established the following laws.

I. *The rotation of the plane of polarization produced by a plate of quartz cut perpendicularly to the axis, is directly proportional to the thickness of the plate, and inversely proportional to the square of the wave-length of the particular light employed.*

II. *If an observer looks along the direction in which the light is travelling, there are certain varieties of quartz which rotate the plane of polarization towards his right hand, whilst there are others which rotate it towards his left hand.*

The former class of crystals are called right-handed, and the latter left-handed.

[1] *Mém. de la prem. classe de l'Inst.* vol. XII. p. 93; see also *Œuvres Complètes* x., p. 36.

[2] *Mém. de l'Acad. des Sciences*, vol. II. p. 41.

From this definition it follows, that if an observer, who is looking through a Nicol's prism at a ray of plane polarized light, places the Nicol in the position of extinction, and then inserts a plate of right-handed quartz, he must turn the Nicol towards his left-hand in order to bring it into the position of extinction; whilst if the plate of quartz is left-handed, he must turn the Nicol towards his right.

Many continental writers adopt a definition, according to which, a plate of quartz is considered *right-handed*, when the observer has to turn his Nicol towards the *right*, in order to bring it into the position of extinction; but I have decided after much consideration, to adopt a definition, in which the directions of propagation and rotation are related in the same manner, as the magnetic force produced by an electric current circulating round the ray.

147. Since the rotation varies inversely as the square of the wave-length, the position of the plane of polarization will be different for different colours. If sunlight be employed, the different colours will be superposed on emergence, and the emergent light will be white; but if the emergent light be examined by a Nicol's prism, placed so that its principal section is parallel to the plane of polarization of any particular colour, that colour will be extinguished, and the light on emergence from the Nicol will appear coloured. The following table gives the values found by Broch[1], for the rotations of the principal lines of the spectrum, produced by a plate of quartz one millimetre in thickness.

Rays	Rotations	Product of rotation multiplied by λ^2
B	15° 18′	7238
C	17° 15′	7429
D	21° 40′	7511
E	27° 28′	7596
F	32° 30′	7622
G	42° 12′	7842

From this table it appears that the law, that the rotation is inversely proportional to the square of the wave-length, is only approximate.

[1] *Ann. de Chim. de la Phys.* (3), xxxiv. p. 119.

148. On account of the smallness of the wave-length of all visible parts of the spectrum, it follows from the first law, that the rotation will amount to a large number of right angles, if the thickness of the plate is considerable. Unless therefore the plate is sufficiently thin for the rotation to amount to less than 180°, the observer is liable to be mistaken as to whether the direction of rotation is to the right or to the left. In fact it appears from the table, that for the line *G* of the spectrum, a plate of quartz only one millimetre in thickness produces a rotation of 42° 12′. By employing plates of different thicknesses all cut from the same specimen of crystal, or by employing light of different colours, all chances of error can be eliminated.

149. The photogyric properties of quartz depend upon the angle which the ray traversing the crystal makes with the axis; they are most marked when the ray is parallel, and disappear when the ray is perpendicular to the axis; for the latter direction, quartz behaves in the same manner as Iceland spar. There are also certain liquids, such as oil of turpentine, essence of lemon, common syrup &c., which possess the power of rotating the plane of polarization of light; and this property is independent of the direction of the transmitted light.

150. Faraday[1] discovered, that when plane polarized light is transmitted through a transparent diamagnetic medium, which is placed in a field of magnetic force, whose direction is parallel to that of the ray, a rotation of the plane of polarization takes place; and the direction of rotation is the same as that in which a positive electric current must circulate round the ray, in order to produce a magnetic force in the same direction as that which actually exists in the medium.

It was afterwards discovered by Verdet[2], that certain ferromagnetic media, such as a strong solution of perchloride of iron in wood spirit, produce a rotation in the opposite direction to that of the current which would give rise to the magnetic force.

151. There is an important distinction between the rotation produced by quartz, turpentine &c., and that produced by a magnetic field. If, when a ray is transmitted through quartz in a given direction, the rotation is from left to right, it is found that

[1] *Experimental Researches*, XIXth series, §§ 2146—2242.

[2] *C. R.* vol. LVI. p. 630; vol. LVII. p. 670; *Ann. de Chim. et de Phys.* (3), vol. LXIX. 415; *Mém. de l'Inst.* vol. XXXI. pp. 106, 341.

when the ray is transmitted in the opposite direction, the rotation is from right to left. If therefore the ray after passing through a plate of quartz, be reflected at perpendicular incidence by a mirror, and thus be made to return through the plate in the same direction, the rotation will be reversed, so that on emerging a second time from the plate, the plane of polarization will be restored to its original position. But when the ray is transmitted through a magnetic field, the direction of rotation in space is always the same, whether the ray is propagated along the positive or negative direction of the magnetic force; if therefore the ray be reflected, and be made to return through the magnetic field, the rotation will be doubled.

The photogyric properties of a magnetic field will be more fully considered, when we discuss the electromagnetic theory of light.

152. The dynamical theories which have been proposed to account for rotatory polarization will be considered later on. At present we shall show how these phenomena can be explained by geometrical considerations.

Fresnel assumed, that the only kind of waves, which media of this class are capable of propagating without change of type, are circularly polarized waves. If light polarized in any other manner is incident upon the medium, Fresnel supposed that the wave is immediately split up, on entering the medium, into two waves which are circularly polarized in opposite directions, and are transmitted with different velocities.

Let us therefore suppose, that a plane polarized wave is incident normally at a point O upon a plate of quartz cut perpendicularly to the axis. Let the incident displacement be parallel to y and equal to $a \sin \phi$, where $\phi = 2\pi t/\tau$; and let the axis of z be the axis of the quartz.

The incident wave may be conceived to be made up of the four displacements

$$u = \tfrac{1}{2} a \cos \phi, \quad u' = -\tfrac{1}{2} a \cos \phi,$$
$$v = \tfrac{1}{2} a \sin \phi, \quad v' = \tfrac{1}{2} a \sin \phi.$$

By § 13, the displacements (u, v) represent a right-handed circularly polarized wave, while the displacements (u', v') represent a left-handed wave. Their combination is equivalent to a single plane polarized wave.

Let d be the thickness of the plate; V_1, V_2 the velocities of propagation of the two waves. Then on emergence, the two waves are represented by

$$u_1 = \tfrac{1}{2}a\cos\frac{2\pi}{\tau}\left(t - \frac{d}{V_1}\right), \quad v_1 = \tfrac{1}{2}a\sin\frac{2\pi}{\tau}\left(t - \frac{d}{V_1}\right),$$

$$u_2 = -\tfrac{1}{2}a\cos\frac{2\pi}{\tau}\left(t - \frac{d}{V_2}\right), \quad v_2 = \tfrac{1}{2}a\sin\frac{2\pi}{\tau}\left(t - \frac{d}{V_2}\right);$$

accordingly the displacement on emergence becomes

$$u = u_1 + u_2 = a\sin\frac{2\pi}{\tau}\left\{t - \tfrac{1}{2}d\left(\frac{1}{V_1} + \frac{1}{V_2}\right)\right\}\sin\frac{\pi d}{\tau}\left(\frac{1}{V_1} - \frac{1}{V_2}\right),$$

$$v = v_1 + v_2 = a\sin\frac{2\pi}{\tau}\left\{t - \tfrac{1}{2}d\left(\frac{1}{V_1} + \frac{1}{V_2}\right)\right\}\cos\frac{\pi d}{\tau}\left(\frac{1}{V_1} - \frac{1}{V_2}\right),$$

whence
$$\frac{u}{v} = \tan\frac{\pi d}{\tau}\left(\frac{1}{V_1} - \frac{1}{V_2}\right)$$

We therefore see that the emergent light is plane polarized, and that the plane of polarization is rotated through an angle

$$\frac{\pi d}{\tau}\left(\frac{1}{V_1} - \frac{1}{V_2}\right)\quad\text{..........................(1).}$$

Hence according as $V_2 >$ or $< V_1$, the rotation will be towards the right-hand or the left-hand of a person who is looking along the positive direction of the axis of z, which is the direction in which the wave is supposed to be travelling. From this result, coupled with the definition in § 146, it follows that in right-handed quartz, the velocity of the left-handed circularly polarized wave is greater than that of the right-handed wave; whilst the converse is the case with left-handed quartz.

Equation (1) is in accordance with the experimental fact, that the rotation is proportional to the thickness, but it does not give any information respecting the dependance of the rotation upon the wave-length, inasmuch as the relation between V_1, V_2 and τ is unknown. Experiment however shows that $V_1 - V_2$ must be a function of the period.

153. From the table on page 158, it follows that a plate of quartz one millimetre in thickness rotates the plane of polarization of the mean yellow rays (that is the rays midway between the lines D and E), through an angle of about 24°. Hence a plate

whose thickness is 15 mm. produces a rotation equal to 360°; and therefore for a plate of quartz of this thickness

$$15V\left(\frac{1}{V_1}-\frac{1}{V_2}\right)=2\lambda \quad \text{.....................(2)},$$

where V is the velocity and λ is the wave-length in air.

If the plate is right-handed, $V_2 > V_1$; also since the difference between the velocities is small, we may put

$$V_2 = V_1 + h,$$

where h is a small quantity, and therefore

$$\frac{V}{V_2}=\frac{V}{V_1}-\frac{Vh}{V_1^2},$$

approximately. Accordingly (2) becomes

$$15Vh/V_1^2\lambda = 2.$$

The ratio V/V_1 is sensibly equal to the ordinary index of refraction of quartz, which is 1·555 or about $\frac{3}{2}$; whence

$$h/V_1 = \tfrac{4}{45}\lambda.$$

For the mean yellow rays, $\lambda = \frac{1}{2} \times 10^{-3}$ mm.; whence

$$h/V_1 = \tfrac{4}{9} \times 10^{-4};$$

from which it appears, that the difference between the two velocities is less than the twenty thousandth part of one of the velocities of the two circularly polarized waves; and that the difference between the two indices of refraction is less than 1·00005, which is less than the index of refraction of the least refrangible gases.

Theory of Coloured Rings.

154. Fresnel confined his attention to the case of plane polarized light transmitted through a plate of quartz in a direction parallel to the axis. The refraction of polarized light, transmitted in an oblique direction, was first studied by Airy[1], whose investigations we shall proceed to consider.

When a plate of quartz is cut parallel to the axis, and the plane of incidence is perpendicular to the latter, Airy found that no appreciable difference existed between the action of quartz, and

[1] *Trans. Camb. Phil. Soc.*, Vol. II. pp. 79, 198; see also Verdet, *Leçons d'Optique Physique*, Vol. II. pp. 237—265.

that of any other uniaxal crystal; but when the plate is cut perpendicularly to the axis, and the incident rays are inclined at a small angle to the axis, he found that the two refracted rays were elliptically polarized in opposite directions. Quartz is a positive crystal; and if the plate is right-handed, it follows from § 13, and also from the definition in § 146, that the left-handed elliptically polarized ray corresponds to the ordinary ray, and the right-handed one to the extraordinary ray. The converse is the case when the plate is left-handed.

The elliptic polarization is only sensible, provided the angle of incidence does not much exceed 10°; if it is considerably greater, the refracted light appears to be plane polarized, and the position of the planes of polarization is the same as in the case of ordinary uniaxal crystals. It therefore follows, that the ratio of the major and minor axes of the two ellipses, which is equal to unity at perpendicular incidence, increases rapidly with the angle of incidence; so that when the latter ceases to be small, the two ellipses do not sensibly differ from two straight lines at right angles to one another. It also follows, that the major axis of the elliptic vibrations of the ordinary ray, is *perpendicular* to the plane containing the optic axis and the ordinary wave normal, whilst in the case of the extraordinary ray, the major axis lies *in* the plane containing the optic axis and the extraordinary wave normal.

We have already shown, that the difference between the ordinary and extraordinary indices of refraction is very small; it was therefore assumed by Airy, that the two ellipses corresponding to the same incident ray are always similar.

When light is transmitted parallel to the axis through an ordinary uniaxal crystal, the velocity of propagation of the two waves is the same; this arises from the fact that the wave surface consists of a sphere and a spheroid, the former of which touches the latter at the extremities of its polar axis. But this result does not hold good in the case of quartz. Airy therefore assumed, that the wave surface in quartz consists of a sphere and a spheroid, which do not touch one another; and since quartz is a positive crystal, the spheroid is prolate, and consequently the spherical sheet lies wholly outside the spheroidal sheet. From this hypothesis, it follows that the difference of path of the two elliptically polarized waves is equal to the corresponding expression for

ordinary uniaxal crystals, together with a certain quantity, which is independent of the angle of incidence, but which is a function of the period.

155. By the aid of the foregoing hypotheses, Airy succeeded in giving a mathematical explanation of the rings and brushes which are produced, when polarized light is passed through a plate of quartz; but before we consider his investigations, it will be desirable to give a description of the peculiarities of these rings.

I. When the planes of polarization and analysation are perpendicular, and a thin plate of quartz is placed between them, a set of circular coloured rings is observed. The centre of the pattern consists of a coloured circular area, and the colour depends upon the thickness of the plate. If the thickness is ·48 of an inch, the central tint is pale pink; with thicknesses of ·38, ·26 and ·17 of an inch, the colours of the central spot are a bright yellowish green, a rich red plum colour, and a rich yellow. The colours of the successive rings, beginning from the centre, appear to be nearly the same as in Newton's scale, commencing with the colour of the central spot. At a considerable distance from the centre, four faint brushes commence, which intersect the rings in the same directions as the black cross in Iceland spar.

II. When the polarizer and analyser are initially crossed, and the latter is made to rotate, a blueish short-armed cross appears in the centre, which on continuing to turn the analyser becomes yellow, and the rings become enlarged. When the inclination of the planes of polarization and analysation is 45°, the rings are nearly square, and the diagonals of the square bisect the angles between these planes.

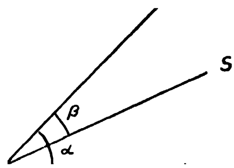
III. When incident light is circularly polarized, the rings consist of two spirals mutually intertwining one another.

IV. When two plates of quartz of equal thickness, one of which is right-handed and the other left-handed, are attached together, and placed between the polarizer and analyser, four spirals proceeding from the black cross to the centre make their appearance. These curves are usually known as Airy's spirals.

156. We shall now proceed to give the mathematical investigation of these phenomena.

We shall suppose, that a small pencil of convergent light is incident upon the crystal at a small angle. Let O be the point of

incidence of any one of the rays, OA the intersection of the plane passing through the ray and the optic axis, at the point where the central ray of the pencil meets the crystal; also let OP, OS be the principal sections of the polarizing and analysing Nicols.



Let $\phi = 2\pi t/\tau$, and let the incident vibration be $\sin\phi$.

If we suppose the crystal right-handed, the incident wave will be resolved into a right-handed elliptically polarized wave, which corresponds to the extraordinary wave, and a left-handed elliptically polarized wave, which corresponds to the ordinary wave; also since the two ellipses of vibration are similar, and their major axes are at right angles, it follows that if (u, v) and (u', v') respectively denote the displacements parallel to OA, OB in the two waves, and k is a quantity lying between zero and unity, we may put

$$\left.\begin{aligned} u &= mk\cos(\phi+\mu), & v &= m\sin(\phi+\mu) \\ u' &= -nk^{-1}\cos(\phi+\nu), & v' &= n\sin(\phi+\nu) \end{aligned}\right\} \quad\text{.........} (3).$$

The resultant of these four displacements must be equal to $\sin\phi$, and must be parallel to OP, whence

$$(u+u')\cos\alpha + (v+v')\sin\alpha = \sin\phi$$

$$(u+u')\sin\alpha - (v+v')\cos\alpha = 0.$$

These equations must hold good for all values of t and therefore of ϕ; whence equating the coefficients of $\sin\phi$ and $\cos\phi$, we obtain

$$(m\cos\mu + n\cos\nu)\sin\alpha - (mk\sin\mu - nk^{-1}\sin\nu)\cos\alpha = 1,$$

$$(m\sin\mu + n\sin\nu)\sin\alpha + (mk\cos\mu - nk^{-1}\cos\nu)\cos\alpha = 0,$$

$$(m\cos\mu + n\cos\nu)\cos\alpha + (mk\sin\mu - nk^{-1}\sin\nu)\sin\alpha = 0,$$

$$(m\sin\mu + n\sin\nu)\cos\alpha + (mk\cos\mu - nk^{-1}\cos\nu)\sin\alpha = 0.$$

From these equations we find, that

$$m\cos\mu + n\cos\nu = \sin\alpha, \qquad mk\cos\mu - nk^{-1}\cos\nu = 0,$$

$$m\sin\mu + n\sin\nu = 0, \qquad mk\sin\mu - nk^{-1}\sin\nu = -\cos\alpha;$$

accordingly

$$m\cos\mu=\frac{\sin\alpha}{1+k^2},\qquad m\sin\mu=-\frac{k\cos\alpha}{1+k^2},$$

$$n\cos\nu=\frac{k^2\sin\alpha}{1+k^2},\qquad n\sin\nu=\frac{k\cos\alpha}{1+k^2}.$$

Substituting in (3), we obtain

$$u=\frac{k}{1+k^2}(\sin\alpha\cos\phi+k\cos\alpha\sin\phi),$$

$$v=\frac{1}{1+k^2}(\sin\alpha\sin\phi-k\cos\alpha\cos\phi),$$

$$u'=-\frac{1}{1+k^2}(k\sin\alpha\cos\phi-\cos\alpha\sin\phi),$$

$$v'=\frac{k}{1+k^2}(k\sin\alpha\sin\phi+\cos\alpha\cos\phi).$$

The first pair of these equations represents the extraordinary wave, and the second pair the ordinary wave.

If D and G are the equivalent paths in air for the ordinary and the extraordinary waves, and if we put

$$\delta=2\pi D/\lambda,\quad \gamma=2\pi G/\lambda;$$

it follows, that on emerging from the crystal, we must replace ϕ by $\phi-\delta$ for the ordinary wave, and by $\phi-\gamma$ for the extraordinary. Hence on emerging from the analyser, the displacements become

$$(u+u')\cos(\alpha-\beta)+(v+v')\sin(\alpha-\beta),$$

or

$$\begin{aligned}(1+k^2)^{-1}\,[\{&k\sin\alpha\cos(\phi-\gamma)+k^2\cos\alpha\sin(\phi-\gamma)-k\sin\alpha\cos(\phi-\delta)\\ &+\cos\alpha\sin(\phi-\delta)\}\cos(\alpha-\beta)\\ +\{&\sin\alpha\sin(\phi-\gamma)-k\cos\alpha\cos(\phi-\gamma)+k^2\sin\alpha\sin(\phi-\delta)\\ &+k\cos\alpha\cos(\phi-\delta)\}\sin(\alpha-\beta)].\end{aligned}$$

Writing $\phi-\gamma+\gamma-\delta$ for $\phi-\delta$, picking out the coefficients of $\cos(\phi-\gamma)$ and $\sin(\phi-\gamma)$, and then squaring and adding, and writing for brevity ψ and χ for $\alpha-\beta$ and $\gamma-\delta$, the intensity I^2 is determined by the equation

$$\begin{aligned}(1+k^2)^2I^2=&\{(k\sin\alpha-k\sin\alpha\cos\chi+\cos\alpha\sin\chi)\cos\psi\\ &+(-k\cos\alpha+k^2\sin\alpha\sin\chi+k\cos\alpha\cos\chi)\sin\psi\}^2\\ &+\{(k^2\cos\alpha+k\sin\alpha\sin\chi+\cos\alpha\cos\chi)\cos\psi\\ &+(\sin\alpha+k^2\sin\alpha\cos\chi-k\cos\alpha\sin\chi)\sin\psi\}^2\\ =&\{\sin\chi(\cos\alpha\cos\psi+k^2\sin\alpha\sin\psi)-k\cos\chi\sin\beta+k\sin\beta\}^2\end{aligned}$$

$$+\{\cos\chi(\cos\alpha\cos\psi+k^2\sin\alpha\sin\psi)+k\sin\chi\sin\beta$$
$$+\sin\alpha\sin\psi+k^2\cos\alpha\cos\psi\}^2$$
$$=(\cos\alpha\cos\psi+k^2\sin\alpha\sin\psi)^2+2k^2\sin^2\beta$$
$$+(\sin\alpha\sin\psi+k^2\cos\alpha\cos\psi)^2$$
$$+2(\cos\alpha\cos\psi+k^2\sin\alpha\sin\psi)\times$$
$$\{\cos\chi(\sin\alpha\sin\psi+k^2\cos\alpha\cos\psi)+k\sin\chi\sin\beta\}$$
$$+2k\sin\chi\sin\beta(\sin\alpha\sin\psi+k^2\cos\alpha\cos\psi)-2k^2\cos\chi\sin^2\beta.$$

Replacing $\cos\chi$ by $1-2\sin^2\frac{1}{2}\chi$, this becomes

$$=(1+k^2)^2\cos^2\beta+k(1+k^2)\sin 2\beta\sin\chi+4k^2\sin^2\tfrac{1}{2}\chi\sin^2\beta$$
$$-4\sin^2\tfrac{1}{2}\chi(\cos\alpha\cos\psi+k^2\sin\alpha\sin\psi)(\sin\alpha\sin\psi+k^2\cos\alpha\cos\psi)$$
$$=(1+k^2)^2\cos^2\beta+k(1+k^2)\sin 2\beta\sin\chi-4k^2\cos 2\beta\sin^2\tfrac{1}{2}\chi$$
$$-(1-k^2)^2\sin 2\alpha\sin 2\psi\sin^2\tfrac{1}{2}\chi,$$

whence dividing by $(1+k^2)^2$, the value of I^2 may be put into the form

$$I^2=\left(\cos\beta\cos\tfrac{1}{2}\chi+\frac{2k}{1+k^2}\sin\beta\sin\tfrac{1}{2}\chi\right)^2$$
$$+\left(\frac{1-k^2}{1+k^2}\right)^2(\cos^2\beta-\sin 2\alpha\sin 2\psi)\sin^2\tfrac{1}{2}\chi.$$

Restoring the values of ψ and χ, this finally becomes

$$I^2=\left\{\cos\beta\cos\tfrac{1}{2}(\delta-\gamma)-\frac{2k}{1+k^2}\sin\beta\sin\tfrac{1}{2}(\delta-\gamma)\right\}^2$$
$$+\left(\frac{1-k^2}{1+k^2}\right)^2\cos^2(2\alpha-\beta)\sin^2\tfrac{1}{2}(\delta-\gamma)\ldots\ldots(4).$$

157. Returning to § 155, we see that the first case which has to be examined, arises when the Nicols are crossed, so that $\beta=\frac{1}{2}\pi$; in this case (4) becomes

$$I^2=\left\{\frac{4k^2}{(1+k^2)^2}+\left(\frac{1-k^2}{1+k^2}\right)^2\sin^2 2\alpha\right\}\sin^2\tfrac{1}{2}(\delta-\gamma)\ \ldots\ (5).$$

In the neighbourhood of the centre of the field of view, k is nearly equal to unity, and the preceding expression shows, that I^2 is very nearly independent of α; hence the centre of the field consists of a bright patch. As we proceed from the centre, k rapidly diminishes to zero, and therefore at some distance from the centre, the intensity is approximately equal to

$$\sin^2 2\alpha\sin^2\tfrac{1}{2}(\gamma-\delta),$$

and therefore vanishes when $\alpha=\frac{1}{2}n\pi$. It therefore follows, that at a certain distance from the centre, four dark brushes make their

appearance, which divide the field into quadrants, but which do not extend right up to the centre; also since the intensity is not absolutely, but only approximately, zero when $\alpha = \frac{1}{2}n\pi$, the brushes are much fainter than in the case of Iceland spar.

We shall now ascertain the form of the rings, which are determined from the condition that

$$\delta - \gamma = 2n\pi,$$

or

$$D - G = n\lambda.$$

If the plate were an ordinary uniaxal crystal, it follows from § 133 that we should have

$$D - G = \frac{T(b^2 - a^2)\,r^2}{2avd^2},$$

where T is the thickness of the plate; and therefore the retardation vanishes at the centre. We know that this is not the case with quartz, and Airy therefore assumed, that the retardation is equal to the sum of this expression, together with a quantity, which is directly proportional to the thickness of the plate and inversely proportional to the wave-length in air. We therefore put

$$D - G = \frac{T}{2av}\{(b^2 - a^2)\,r^2 + H/\lambda\} \quad \text{...........} \quad (6),$$

where H is a constant.

In the neighbourhood of the centre, the intensity is very nearly equal to

$$\sin^2 \pi(D - G)/\lambda,$$

and since this expression does not vanish when $r = 0$, the centre can never be black; also since $\pi(D - G)/\lambda$ is of the form $(A/\lambda + B/\lambda^2)\,T$, it follows that the colour of the central spot will vary with the thickness of the plate.

At some distance from the centre, k will be small; whence the intensity of the rings will be a maximum or minimum according as

$$D - G = (n + \tfrac{1}{2})\lambda \text{ or } n\lambda;$$

accordingly the isochromatic curves are circles.

The pattern therefore consists of a bright coloured circular spot in the centre of the field, surrounded by coloured circular rings; and the rings are interrupted by four faint brushes at right angles to one another, which commence at the circumference of the circular spot.

158. If $\beta = 0$, so that the planes of polarization and analysation are parallel, the expression for the intensity becomes

$$1 - \left\{\frac{4k^2}{(1+k^2)^2} + \left(\frac{1-k^2}{1+k^2}\right)^2 \sin^2 2\alpha\right\} \sin^2 \tfrac{1}{2}(\delta - \gamma),$$

from which it appears that the rings are interrupted by two white brushes, and that the colours are complementary to those in the former case.

The forms of the curves are shown in figures 5 and 6 of the plate at the end of this Chapter.

159. When the planes of polarization and analysation are neither parallel nor perpendicular to one another, the isochromatic curves are of a more complicated character. In order to get an approximate idea of their form, let

$$\frac{2k \tan\beta}{1+k^2} = \tan\psi.$$

Substituting the values of $\sin\beta$, $\cos\beta$ deduced from this equation in the first term of (4), it becomes

$$\frac{4k^2}{4k^2 + (1+k^2)\tan^2\psi}\{\cos\tfrac{1}{2}(\delta-\gamma) - \tan\psi \sin\tfrac{1}{2}(\delta-\gamma)\}^2$$

$$= \left\{\cos^2\beta + \frac{4k^2}{(1+k^2)^2}\sin^2\beta\right\}\cos^2\tfrac{1}{2}(\delta - \gamma + 2\psi),$$

whence

$$I^2 = \left\{\cos^2\beta + \frac{4k^2}{(1+k^2)^2}\sin^2\beta\right\}\cos^2\tfrac{1}{2}(\delta-\gamma+2\psi)$$

$$+ \left(\frac{1-k^2}{1+k^2}\right)^2 \cos^2(2\alpha-\beta)\sin^2\tfrac{1}{2}(\delta-\gamma)\ldots\ldots\ldots\ldots(7).$$

If we suppose that a small variation of $\delta - \gamma$ does not produce any sensible alteration in the value of k, the maximum or minimum value of the intensity for given values of α and β will be found by differentiating I^2 with respect to $\delta - \gamma$, on the supposition that k is constant; we thus obtain

$$\tan(\delta - \gamma + \psi)$$

$$= -\tan\psi\,\frac{(1+k^2)^2\cos^2\beta + 4k^2\sin^2\beta + (1-k^2)^2\cos^2(2\alpha-\beta)}{(1+k^2)^2\cos^2\beta + 4k^2\sin^2\beta - (1-k^2)^2\cos^2(2\alpha-\beta)}\ldots(8)$$

$$= -\tan\Omega \text{ (say).}$$

Let Q be any point of the coloured image, QP the direction of vibration of the incident light, QS the principal section of the analyser; then if O is the centre of the field, OQ is the principal

section corresponding to the ray Q. Now as we proceed along OQ, the intensity depends upon the value of $\delta - \gamma$; and we see from (8), that the points of maximum and minimum brightness occur, when $\delta - \gamma$ has a value which depends upon α and k. Now if $OQ = r$, it follows from (6), that

$$\delta - \gamma = Ar^2 + B,$$

where A and B are constants. Hence as OQ revolves around the origin, the points of maximum and minimum brightness will not be equidistant from O, but will lie on a sort of square curve.

The equation of these curves is

$$Ar^2 = \pi - \Omega - \psi - B.$$

Now Ω is a maximum when $\alpha = \frac{1}{2}\beta + \frac{1}{2}n\pi$, and a minimum when $\alpha = \frac{1}{2}\beta + (\frac{1}{2}n + \frac{1}{4})\pi$; and therefore Ω is a maximum when OQ bisects the interior and exterior angles between the planes of polarization and analysation, and is a minimum when OQ makes an angle $\frac{1}{4}\pi$ with its four preceding positions. The best position of the analyser for viewing these curves is when $\beta = \frac{1}{4}\pi$, in which case the isochromatic curves form a sort of square, whose diagonals bisect the angles between the planes of polarization and analysation.

In the central portions of the field, k is very nearly equal to unity, and from (7) it appears, that the intensity will be least when $\delta - \gamma + 2\psi = (2n + 1)\pi$; consequently there will be a dark spot in the central portion. Now for points equidistant from the centre, $\delta - \gamma$ has the same value; and we see from (7), that for all points which are near the centre, the intensity is approximately greatest when $\alpha = \frac{1}{2}\beta + \frac{1}{2}n\pi$, and least when $\alpha = \frac{1}{2}\beta + (\frac{1}{2}n + \frac{1}{4})\pi$; hence in the centre of the field there is a dark cross, whose arms coincide with the diagonals of the square curves.

160. We shall in the next place investigate the rings and brushes which are produced when the incident light is circularly polarized, and we shall suppose that the plate and the polarization are right-handed.

Let the incident light be

$$u = \cos\phi, \quad v = \sin\phi;$$

where $\phi = 2\pi t/\tau$.

On entering the quartz at oblique incidence, the incident light is resolved into two elliptically polarized waves, one of which is

right-handed and the other left-handed; and therefore within the crystal we may put

$$u_1 = mk\cos(\phi+\mu), \qquad v_1 = m\sin(\phi+\mu),$$
$$u_2 = -nk^{-1}\cos(\phi+\nu), \quad v_2 = n\sin(\phi+\nu).$$

Since these two sets of vibrations are equivalent to the incident vibrations, we must have

$$u = u_1 + u_2, \quad v = v_1 + v_2;$$

whence equating coefficients of $\cos\phi$, $\sin\phi$, we get

$$mk\cos\mu - nk^{-1}\cos\nu = 1,$$
$$mk\sin\mu - nk^{-1}\sin\nu = 0,$$
$$m\sin\mu + n\sin\nu = 0,$$
$$m\cos\mu + n\cos\nu = 1,$$

from which we deduce

$$\mu = 0, \qquad \nu = 0.$$
$$m = \frac{1+k}{1+k^2}, \qquad n = -\frac{k(1-k)}{1+k^2}.$$

Hence, on emerging from the crystal, the two rays are represented by the equations

$$u_1 = \frac{k(1+k)}{1+k^2}\cos(\phi-\gamma), \qquad v_1 = \frac{1+k}{1+k^2}\sin(\phi-\gamma),$$
$$u_2 = \frac{1-k}{1+k^2}\cos(\phi-\delta), \qquad v_2 = -\frac{k(1-k)}{1+k^2}\sin(\phi-\delta).$$

If α be the angle which the plane analysation makes with the principal section of the crystal, the light on emerging from the crystal is represented by

$$(u_1+u_2)\cos\alpha + (v_1+v_2)\sin\alpha;$$

that is $$\frac{\cos\alpha}{1+k^2}\{k(1+k)\cos(\phi-\gamma) + (1-k)\cos(\phi-\delta)\},$$
$$+\frac{\sin\alpha}{1+k^2}\{(1+k)\sin(\phi-\gamma) - k(1-k)\sin(\phi-\delta)\}.$$

Replacing $\phi-\delta$ by $\phi-\gamma+\gamma-\delta$, and putting χ for $\gamma-\delta$, the intensity is determined by the equation

$$\begin{aligned}(1+k^2)^2 I^2 &= [\{1+k-k(1-k)\cos\chi\}\sin\alpha - (1-k)\sin\chi\cos\alpha]^2,\\ &+ [\{(1-k)\cos\chi + k(1+k)\}\cos\alpha - k(1-k)\sin\chi\sin\alpha]^2\\ &= \{(1+k)^2 - 2k(1-k^2)\cos\chi + k^2(1-k)^2\}\sin^2\alpha\\ &+ \{(1-k)^2 + 2k(1-k^2)\cos\chi + k^2(1+k)^2\}\cos^2\alpha\\ &- (1-k^4)\sin 2\alpha\sin\chi,\end{aligned}$$

whence

$$I^2 = 1 - \frac{2k(1-k^2)}{(1+k^2)^2}\cos 2\alpha$$
$$+ \frac{2k(1-k^2)}{(1+k^2)^2}\cos 2\alpha \cos\chi - \frac{1-k^2}{1+k^2}\sin 2\alpha \sin\chi \quad\text{.........(9).}$$

161. If in this expression we put

$$(1+k^2)\tan 2\alpha = 2k\tan 2\psi$$

and restore the value of χ, we obtain

$$I^2 = 1 - \frac{2k(1-k^2)}{(1+k^2)^2}\cos 2\alpha + \frac{2k(1-k^2)\cos(\delta-\gamma-2\psi)}{(1+k^2)\{(1+k^2)^2\cos^2 2\psi + 4k^2\sin^2 2\psi\}^{\frac{1}{2}}} \quad\text{...(10).}$$

Let us now draw a line from the centre making an angle α with the principal section of the analyser. Then if we consider a series of points on this line, which are not very distant from one another, we may suppose that k is approximately constant for such points. From (10) we see that I^2 is a maximum or minimum according as

$$\delta - \gamma - 2\psi = 2n\pi \text{ or } (2n+1)\pi;$$

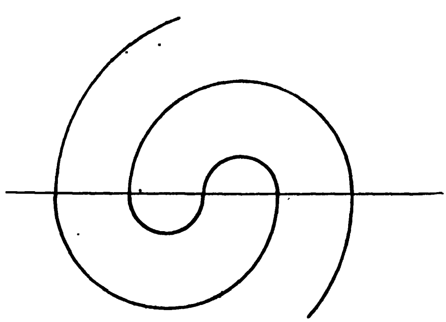
and since $\qquad \delta - \gamma = Ar^2 + B,$

the points of maximum intensity are determined by

$$Ar^2 + B = 2n\pi + 2\psi \quad\text{....................(11).}$$

In the neighbourhood of the centre, k does not differ much from unity, and we may therefore as a first approximation put $\psi = \alpha$; whence writing θ for α, the equation of the isochromatic curves becomes

$$Ar^2 = 2n\pi + 2\theta - B.$$



This equation represents a spiral curve which commences at the origin. The form of the curve when $n=1$ is shown in the figure; if we put $n=2$, we obtain a second spiral which is derived

from the former by turning it through two right angles. For values of n greater than two, the two spirals will be found to be reproduced.

162. The fourth case which we shall consider, arises when plane polarized light is incident upon two plates of quartz of equal thickness, one of which is right-handed, and the other left-handed; and the planes of polarization and analysation are parallel.

The displacements in the two elliptically polarized waves on emergence from the first plate are given in § 156; and we must recollect that on emergence, we must write $\phi+\chi$ for ϕ in the values of (u', v').

Since the second plate of quartz is left-handed, the sign of k must be reversed, and therefore on entering the second plate we must write

$$U=-mk\cos(\phi+\mu), \qquad V=m\sin(\phi+\mu)$$

for the ordinary wave, and

$$U'=nk^{-1}\cos(\psi+\nu), \qquad V'=n\sin(\psi+\nu)$$

for the extraordinary wave where $\psi=\phi+\chi$.

The four quantities m, n, μ, ν must be determined by equating the coefficients of $\sin\psi$, $\cos\psi$ in the equation

$$u+u'=U+U', \qquad v+v'=V+V'.$$

Having obtained the values of m, n, μ, ν, we must write $\psi+\chi$ for ψ in the expressions for U', V'.

Since the planes of polarization and analysation are supposed to be perpendicular, the displacement on emergence from the analyser will be

$$(U+U')\cos\alpha+(V+V')\sin\alpha;$$

we must therefore form this expression, and then write down the sum of the squares of the coefficients of $\sin\phi$, $\cos\phi$, which will give the intensity.

The actual calculations are somewhat tedious, but on performing the above operations, it will be found that

$$I^2=\left(\frac{1-k^2}{1+k^2}\right)^2\sin^2\tfrac{1}{2}(\delta-\gamma)\times$$
$$\left\{\frac{4k}{1+k^2}\cos 2\alpha\sin\tfrac{1}{2}(\delta-\gamma)-2\sin 2\alpha\cos\tfrac{1}{2}(\delta-\gamma)\right\}$$

163. This expression can vanish in two ways. In the first case $\sin \frac{1}{2}(\delta - \gamma) = 0$, which requires that

$$\delta - \gamma = 2n\pi,$$

which represents a series of circular rings, which are black if homogeneous light be employed, but coloured if white light be used.

In the second case the intensity will vanish when

$$\tan \tfrac{1}{2}(\delta - \gamma) = \frac{1 + k^2}{2k} \tan 2\alpha.$$

In the neighbourhood of the centre, k does not differ much from unity, and we may therefore take as a first approximation

$$\delta - \gamma = 4\alpha + 2n\pi\,;$$

whence writing θ for α, the equation of the isochromatic curves are

$$Ar^2 + B = 4\theta + 2n\pi,$$

which is the equation of a spiral curve.

Let us first suppose that $n = 0$; then it follows that when $\theta = \frac{1}{4}B$, $r = 0$; so that the spiral commences at the origin, and the distances of successive points from the origin increase with θ. When $\theta = \frac{1}{2}\pi$, $r^2 = (2\pi - B)/A$.

Next let $n = 1$, then $r = 0$, when $\theta = \frac{1}{4}B - \frac{1}{2}\pi$; and when $\theta = \frac{1}{4}B$, $r^2 = (2\pi - B)/A$. We therefore see that the spiral corresponding to $n = 1$ is equivalent to the spiral corresponding to $n = 0$, turned from left to right through a right angle.

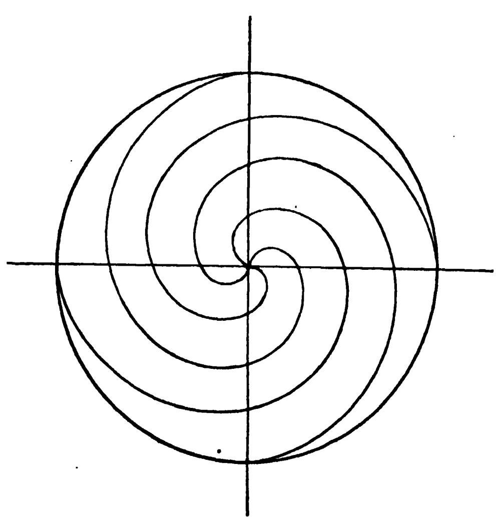


Fig. 1.

Fig.

Fig. 3.

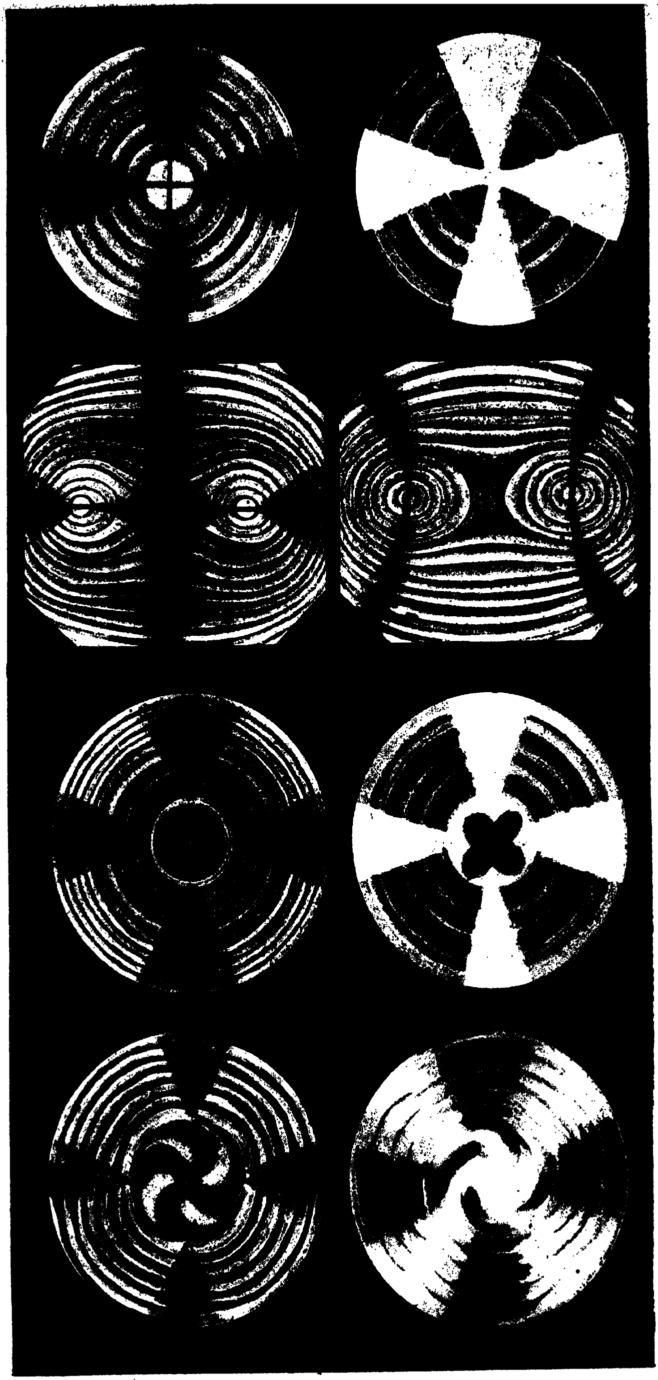
Fig.

Fig. 5.

Fig.

Fig. 7.

Fig.



B. O.

[*To face page* 174.

Similarly for $n = 2$, $n = 3$ there are two other spirals, whose positions can be obtained by turning the spiral for which $n = 0$ backwards through two right angles, and three right angles respectively. When $n = 4$, the original spiral is reproduced.

The forms of these spirals, which after their discoverer are usually known as Airy's Spirals, are shown in the figure on the last page, and also in figures 7 and 8 of the plate. At a considerable distance from the centre, a faint black cross makes its appearance, whose four arms are parallel and perpendicular to the plane of polarization of the incident light; also the spirals disappear and are replaced by circular rings. Now at a distance from the centre, k is nearly equal to zero; whence the intensity becomes

$$I^2 = \sin^2 2\alpha \sin^2 (\delta - \gamma),$$

which vanishes when

$$\gamma - \delta = n\pi$$

or

$$\alpha = \tfrac{1}{2}n\pi.$$

The first equation gives the circular rings, whilst the latter equation gives the brushes.

CHAPTER X.

FRESNEL'S THEORY OF REFLECTION AND REFRACTION.

164. WHEN common light is incident upon the surface of a transparent medium, such as glass, it can be proved experimentally, that the proportion of the incident light which is reflected or refracted, depends upon the angle of incidence; and that the amount of light reflected is greater when the angle of incidence is large, than when it is small. It is also known, that when light proceeding from a denser medium, such as glass, is incident upon a rarer medium, such as air, at an angle greater than the critical angle, the intensity of the reflected light is very nearly equal to that of the incident light, and the reflection is said to be total. When the incident light is polarized *in* the plane of incidence, the effect produced by a reflecting medium is not very different from that produced upon common light; but when the light is polarized perpendicularly to the plane of incidence, it is found that the intensity gradually diminishes from grazing incidence, and very nearly vanishes, when the angle of incidence is equal to $\tan^{-1}\mu$, where μ is the index of refraction of the reflecting substance; as the angle of incidence still further increases, the intensity of the reflected light increases to normal incidence.

165. That the intensity of light polarized perpendicularly to the plane of incidence is zero for a certain angle of incidence, was first discovered by Malus, who while examining with a prism of Iceland spar the light reflected from one of the windows of the Luxembourg palace at Paris, observed that for a certain position of the prism, one of the two images of the sun disappeared. On turning the prism round the line of sight, this image reappeared; and when the prism was turned through 90°, the second image

reappeared. More accurate experiments were afterwards made by Brewster[1], who discovered that when the reflector is an isotropic transparent substance, and the incident light is polarized perpendicularly to the plane of incidence, the intensity of the reflected light is zero, or very nearly so, when the angle of incidence is equal to $\tan^{-1}\mu$. This discovery is known as Brewster's law, and the angle $\tan^{-1}\mu$ is called the polarizing angle.

166. Brewster's law has been tested by Sir John Conroy[2] for transparent bodies in contact with media other than air, in the following manner. A glass prism was placed in contact with water and with carbon tetrachloride respectively, and the polarizing angles were determined. Their values, as found by experiment, were as follows:

Polarizing angle	in air	57° 14′
" "	in water	49° 41′
" "	in carbon tetrachloride	46° 32′.

The polarizing angles were then determined experimentally for water and carbon tetrachloride in contact with air, and the values of the polarizing angles for glass in contact with these substances were then calculated. The results were as follows:

Polarizing angle	in air observed	57° 14′
" "	calculated from observations in water	57° 28′
" "	in carbon tetrachloride	57° 01′.

These results show, that within the limits of experimental error, Brewster's law holds good for glass in contact with water and carbon tetrachloride, as well as air; and that in all probability, it is true for most transparent bodies.

167. Crystalline substances, such as Iceland spar, also possess a polarizing angle as well as a critical angle. In isotropic media, the critical angle is equal to $\sin^{-1}\mu$; but in doubly refracting media, the values of the polarizing and critical angles cannot be so simply expressed.

168. Metallic substances, such as polished silver, possess a quasi-polarizing angle, since there is a particular angle of incidence at which the intensity of light polarized perpendicularly to the plane of incidence is a minimum.

[1] *Phil. Trans.* 1815, p. 125. See also Lord Rayleigh, "On Reflection from Liquid Surfaces in the Neighbourhood of the Polarizing Angle," *Phil. Mag.* Jan. 1892.

[2] *Proc. Roy. Soc.* vol. XXXI. p. 487.

169. In order to explain these experimental facts, it is necessary to determine the intensities of the reflected and refracted lights. This was first effected by Fresnel; and although his theory is not rigorous, it will be desirable to give an account of it in the present Chapter. Other theories based upon speculations respecting the physical constitution of the ether, which are developed according to strict dynamical principles, will be considered in subsequent chapters; and it will be found that most of them give results, which are substantially in accordance with those obtained by Fresnel.

170. We shall first calculate the rate at which energy flows across the reflecting surface.

Let the incident vibration be

$$w = A\cos\frac{2\pi}{\lambda}(x - Vt),$$

and let us consider the energy contained within a small cylinder whose cross section is dS, and whose sides coincide with the direction of propagation. If T denote the amount of kinetic energy per wave-length

$$\begin{aligned} T &= \frac{2\pi^2A^2V^2\rho dS}{\lambda^2}\int_0^\lambda \sin^2\frac{2\pi}{\lambda}(x - Vt)\,dx \\ &= \frac{\pi^2A^2V^2\rho dS}{\lambda}. \end{aligned}$$

Since this amount of kinetic energy flows across dS in time τ, the rate at which kinetic energy flows across dS is $\pi^2A^2V^2\rho dS/\lambda\tau$.

Let dS' be any oblique section of the cylinder, which makes an angle e with dS, then if we assume that the energy of the wave is half kinetic and half potential, the rate at which energy flows across dS' is

$$2\pi A^2V^2\rho\cos e dS'/\lambda\tau \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

The mean energy for unit of volume is

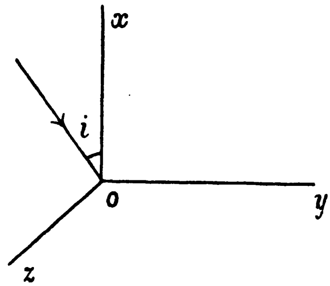
$$2\pi^2A^2V^2\rho/\lambda^2 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

as has already been shown in § 10.

171. We are now prepared to consider the problem of reflection and refraction.

Let the axis of x be normal to the reflecting surface, and let the axis of z be perpendicular to the plane of incidence.

Let the incident light be polarized in the plane of incidence,



then the incident, reflected and refracted waves may be taken to be the real parts of w, w', w_1, where

$$\left.\begin{aligned} w &= A\epsilon^{\iota\kappa(lx+my-Vt)} \\ w' &= A'\epsilon^{\iota\kappa(l'x+m'y-V't)} \\ w_1 &= A_1\epsilon^{\iota\kappa_1(l_1x+m_1y-V_1t)} \end{aligned}\right\}\ldots\ldots\ldots\ldots(3),$$

and $\kappa = 2\pi/\lambda$, $\kappa_1 = 2\pi/\lambda_1$.

Since the reflected and refracted waves are forced vibrations produced and maintained by the incident wave, it follows that the periods of the three waves must be the same; whence

$$\kappa V = \kappa_1 V_1, \text{ or } V/\lambda = V_1/\lambda_1 \ldots\ldots\ldots\ldots(4).$$

Since the traces of all three waves on the plane $x = 0$ move together, it follows that the coefficients of y must be the same in all three waves, whence

$$\kappa m = \kappa m' = \kappa_1 m_1 \ldots\ldots\ldots\ldots(5).$$

If i, i', r be the angles of incidence, reflection and refraction,

$$l = -\cos i, \qquad l' = \cos i', \qquad l_1 = -\cos r,$$

$$m = \sin i, \qquad m' = \sin i', \qquad m_1 = \sin r,$$

and therefore from (5), $i = i'$,

$$\frac{\sin i}{\sin r} = \frac{\kappa_1}{\kappa} = \frac{V}{V_1}$$

which is the law of sines.

172. We now require two equations connecting the amplitudes of the reflected and refracted waves. In order to effect this, Fresnel assumed, (i) *that the displacements at the surface of separation are the same in the two media*, (ii) *that the rate at which energy flows across this surface is continuous.*

The first condition gives

$$A + A' = A_1 \qquad\qquad (6),$$

and by (1) the second condition gives

$$A^2V^2\rho\lambda^{-1}\cos i - A'^2V^2\rho\lambda^{-1}\cos i = A_1^2V_1^2\rho_1\lambda_1^{-1}\cos r$$

or

$$(A^2 - A'^2)\, V\rho\cos i = A_1^2 V_1\rho_1\cos r \qquad\qquad (7).$$

Fresnel's third assumption was, *that the product of the velocity into the square root of the density is constant for all media;* which gives

$$V/V_1 = (\rho_1/\rho)^{\frac{1}{2}}.$$

Accordingly (7) becomes

$$(A^2 - A'^2)\tan r = A_1^2\tan i \qquad\qquad (8).$$

Solving (6) and (7), we obtain

$$\left.\begin{aligned} A' &= -\frac{A\sin(i-r)}{\sin(i+r)} \\ A_1 &= \frac{2A\sin r\cos i}{\sin(i+r)} \end{aligned}\right\} \qquad\qquad (9).$$

But if I, I', I_1 denote the square roots of the intensities, it follows from (3) of § 10, that

$$\frac{I}{A\rho^{\frac{1}{2}}} = \frac{I'}{A'\rho^{\frac{1}{2}}} = \frac{I_1}{A_1\rho_1^{\frac{1}{2}}} \qquad\qquad (10).$$

Accordingly

$$\left.\begin{aligned} I' &= -\frac{I\sin(i-r)}{\sin(i+r)} \\ I_1 &= \frac{I\sin 2i}{\sin(i+r)} \end{aligned}\right\} \qquad\qquad (11).$$

These formulæ give the ratios of the intensities of the reflected and refracted light to that of the incident light.

173. When the second medium is more highly refracting than the first, as is the case when light proceeding from air is reflected at the surface of glass, r is always real; but in the converse case, r is imaginary when the angle of incidence exceeds the critical angle. For if μ be the index of refraction from air to glass, and light is internally reflected and refracted at the surface of glass in contact with air,

$$\sin i = \mu^{-1}\sin r,$$

$$\cos r = (1 - \mu^2\sin^2 i)^{\frac{1}{2}}.$$

Since $\mu > 1$, it follows that $\cos r$ is imaginary when $i > \sin^{-1}\mu^{-1}$. Under these circumstances, the expressions for the amplitudes of

the reflected and refracted waves become complex, and their interpretation in former times was supposed to be a matter of considerable difficulty. The true explanation is this. The incident, reflected and refracted waves are the real parts of the right-hand sides of (3); if therefore A' and A_1 are real, the reflected and refracted waves are

$$w' = A' \cos \kappa (-lx + my - Vt)$$
$$w_1 = A_1 \cos \kappa_1 (l_1 x + m_1 y - V_1 t);$$

but if A' and A_1 are complex, we must write $A' = \alpha + \iota\beta$, $A_1 = \alpha_1 + \iota\beta_1$, and the reflected wave is

$$\begin{aligned} w' &= \alpha \cos \kappa (-lx + my - Vt) - \beta \sin \kappa (-lx + my - Vt) \\ &= (\alpha^2 + \beta^2)^{\frac{1}{2}} \cos \{\kappa (-lx + my - Vt) + \tan^{-1} \beta/\alpha\} \end{aligned}$$

which shows that there is a change of phase.

To find α, β, we have from the first of (9)

$$\alpha + \iota\beta = \frac{A \{\mu \cos i - \iota (\mu^2 \sin^2 i - 1)^{\frac{1}{2}}\}}{\mu \cos i + \iota (\mu^2 \sin^2 i - 1)^{\frac{1}{2}}};$$

whence if $$\tan \frac{\pi e}{\lambda} = \frac{(\mu^2 \sin^2 i - 1)^{\frac{1}{2}}}{\mu \cos i} \qquad (12)$$

we obtain $$\alpha + \iota\beta = A\epsilon^{-2\iota\pi e/\lambda}$$

where λ is the wave-length in glass;

whence $$\tan^{-1}\beta/\alpha = -2\pi e/\lambda.$$

Also $(\alpha^2 + \beta^2)^{\frac{1}{2}} = A$, so that the reflected wave becomes

$$w = A \cos \frac{2\pi}{\lambda} (x \cos i + y \sin i - Vt - e)$$

which shows that the reflection is total, and is accompanied by a change of phase whose value is given by (12).

174. To find what the refracted wave becomes, we have from (9)

$$\alpha_1 + \iota\beta_1 = \frac{2A\mu \cos i}{\mu \cos i + \iota (\mu^2 \sin^2 i - 1)^{\frac{1}{2}}}$$

whence $$\alpha_1^2 + \beta_1^2 = \frac{4\mu^2 \cos^2 i}{\mu^2 - 1}$$

$$\frac{\beta_1}{\alpha_1} = -\frac{(\mu^2 \sin^2 i - 1)^{\frac{1}{2}}}{\mu \cos i}.$$

Also $$\iota\kappa_1 l_1 = -\iota\kappa_1 \cos r = \frac{2\pi}{\lambda\mu} (\mu^2 \sin^2 i - 1)^{\frac{1}{2}},$$

whence the refracted wave is

$$\frac{2A\mu\cos i}{(\mu^2-1)^{\frac{1}{2}}}\,\epsilon^{\frac{2\pi}{\lambda\mu}(\mu^2\sin^2 i-1)^{\frac{1}{2}}x}\cos\frac{2\pi}{\lambda}(y\sin i-Vt-\tfrac{1}{2}e)\ldots\ldots(13).$$

Since x is negative in the second medium, it follows that the wave penetrates only a very short distance, and becomes insensible at a distance of a few wave-lengths[1].

175. The preceding theory is rigorous from a dynamical point of view; but when we consider the corresponding problem in which the incident light is polarized perpendicularly to the plane of incidence, we shall find that a difficulty arises, which will be considered in § 180.

The displacements in the three waves are given by (3), and they lie in the plane of xy and are perpendicular to the direction of propagation of the waves.

The condition that the displacements parallel to y should be continuous gives

$$(A-A')\cos i=A_1\cos r\ldots\ldots\ldots\ldots\ldots\ldots(14).$$

Combining this with (8) we get

$$\left.\begin{aligned}A'&=\frac{A\tan(i-r)}{\tan(i+r)}\\ A_1&=\frac{2A\cos i\sin r}{\sin(i+r)\cos(i-r)}\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots(15),$$

whence

$$\left.\begin{aligned}I'&=\frac{I\tan(i-r)}{\tan(i+r)}\\ I_1&=\frac{I\sin 2i}{\sin(i+r)\cos(i-r)}\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots(16).$$

The first of these formulæ shows, that the intensity of the reflected light vanishes when

$$i+r=\tfrac{1}{2}\pi,\quad\text{or}\quad\tan i=\mu.$$

[1] Another explanation, differing only in form, is as follows. The hypothesis that the reflected wave, corresponding to the incident wave

$$A\cos 2\pi\lambda^{-1}(-x\cos i+y\sin i-Vt),\ \text{is}\ A'\cos 2\pi\lambda^{-1}(x\cos i+y\sin i-Vt),$$

tacitly involves the assumption, that reflection is unaccompanied by a change of phase. The fact that the amplitude becomes complex when the angle of incidence exceeds the critical angle shows, that this assumption is erroneous in this particular case. We ought therefore to assume that a change of phase takes place, both in the reflected and refracted wave; and we shall find, that the changes of phase are zero, when the angle of incidence is less than the critical angle, and have the above values when it exceeds it.

176. When light of any kind is incident upon a transparent reflecting surface, the vibrations may be resolved into two components respectively in and perpendicular to the plane of incidence; and the first of (16) shows, that the component of the reflected vibration *in the plane of incidence* vanishes, when the angle of incidence is equal to $\tan^{-1}\mu$. It therefore follows, that if common light be incident at this angle, the reflected light will be polarized in the plane of incidence. This is the law which was established experimentally by Brewster.

177. Airy observed, that certain highly refracting substances, such as diamond, never completely polarize common light at any angle of incidence, but the proportion of polarized light is a maximum at the polarizing angle. The subject has been further investigated experimentally by Jamin[1], who found that for most transparent substances, Brewster's law is true as a first approximation only. It is therefore not possible to completely polarize light by a single reflection, but this may be accomplished by successive reflections from a pile of plates. Jamin also found, that when light which is plane polarized in any azimuth, is reflected from a transparent substance, the reflected light frequently exhibits slight traces of elliptic polarization; this shows, that reflection produces a difference of phase in one or both of the components of the reflected light[2]. The reflection and refraction of light incident perpendicularly upon a glass plate have been experimentally investigated by Rood[3], and his results show that Fresnel's formulæ are very approximately correct.

178. When light proceeding from glass, is reflected at the surface of a rarer medium such as air, at an angle greater than the critical angle, it will be found that the values of A', A_1 given by (15) become complex; and it can be shown in the same manner

[1] *Ann. de Chimie et de Phys.* (3), XXIX. pp. 31 and 263; *Ibid.* (3), XXX. p. 257.

[2] Owing to the extreme smallness of the wave-length of light, compared with the ordinary standards of measurement, it is probable that if the surface of a polished reflector were magnified to such an extent, that the wave-length of light were represented by one inch, the surface of the reflector would appear to be exceedingly rough and uneven. It is therefore by no means improbable, that the secondary effects observed by Jamin, may be due to the fact, that our mathematical machinery is too coarse-grained to take into account inequalities of the reflecting surface, which though excessively minute, are not small compared with the wave-length of light.

[3] *Amer. Jour. of Science*, vol. I. July 1870.

that the reflection is total, and is accompanied by a change of phase. In fact the reflected wave is

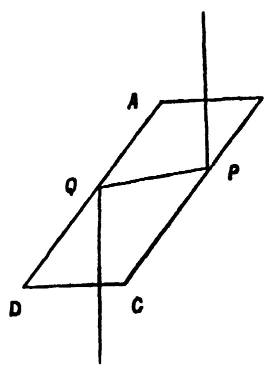
$$w' = A_{,} \cos \frac{2\pi}{\lambda} (x \cos i + y \sin i - Vt - e_1)$$

where $$\tan \frac{\pi e_1}{\lambda} = \frac{\mu (\mu^2 \sin^2 i - 1)^{\frac{1}{2}}}{\cos i} \quad \text{.(17)},$$

and the refracted wave is

$$\frac{2A\mu \cos i}{\{\cos^2 i + \mu^2 (\mu^2 \sin^2 i - 1)\}^{\frac{1}{2}}} \epsilon^{\frac{2\pi}{\lambda\mu} (\mu^2 \sin^2 i - 1)^{\frac{1}{2}} x} \cos \frac{2\pi}{\lambda} (y \sin i - Vt - \tfrac{1}{2} e_1).$$

179. The change of phase which accompanies total reflection, was experimentally verified by Fresnel in the following manner.



Let $ABCD$ be a rhomb of glass, of which the angles at B and D are greater than the critical angle; and let light polarized in a plane which makes an angle of 45° with the plane of incidence (that is the plane of the paper), be incident normally upon the face AB, and after undergoing two reflections emerge at the face DC. The vibrations in and perpendicular to the plane of incidence after emergence, will be represented by

$$A \cos \frac{2\pi}{\lambda} (x - Vt - 2e_1),$$

and $$A \cos \frac{2\pi}{\lambda} (x - Vt - 2e).$$

If therefore $$e_1 - e = \tfrac{1}{8}\lambda,$$

the emergent light will be circularly polarized. Now if

$$\pi (e_1 - e)/\lambda = \delta,$$

we obtain from (12) and (17)

$$\tan \delta = \frac{\cos i\,(\mu^2 \sin^2 i - 1)^{\frac{1}{2}}}{\mu \sin^2 i},$$

whence

$$\cos 2\delta = \frac{1 - \tan^2 \delta}{1 + \tan^2 \delta} = \frac{2\mu^2 \sin^4 i - (1 + \mu^2) \sin^2 i + 1}{(1 + \mu^2) \sin^2 i - 1}.$$

Hence if $\delta = \frac{1}{8}\pi$, this becomes

$$4\mu^2 \sin^4 i - (2 + \sqrt{2})(1 + \mu^2) \sin^2 i + 2 + \sqrt{2} = 0.$$

This equation gives a real value of $\sin i$ for values of μ lying between 1·4 and 1·6.

Fresnel employed a rhomb of St Gobain glass, for which $\mu = 1{\cdot}51$, which gives $i = 48^\circ\ 37'\ 3''$ or $54^\circ\ 37'\ 20''$. Now the angles at B and D of the rhomb are each equal to the angle of incidence; if therefore a rhomb of glass, whose index of refraction is 1·51, and whose acute angles are equal to $54^\circ\ 37'\ 20''$ be employed, and light polarized as described above is incident normally on the face AB and is reflected twice, the emergent light ought to be circularly polarized. This result was found to agree with experiment.

If the incident light is polarized in any other plane, the emergent light will be elliptically polarized.

If α be the azimuth of the plane of polarization, and the emergent elliptically polarized light be passed through a second rhomb, the reflected light will be plane polarized, and the plane of polarization will be rotated through an angle 2α.

Theories of Neumann and MacCullagh[1].

180. We must now consider the difficulty alluded to at the commencement of § 175.

The surface conditions assumed by Fresnel are, (i) continuity of the rate at which energy flows across the reflecting surface, (ii) continuity of the components of displacement parallel to this surface. Now when the incident light is polarized in the plane of incidence, there is no component displacement perpendicular to this surface; but when the light is polarized perpendicularly to

[1] Neumann, *Abhand. Berlin Akad.* 1835.

MacCullagh, "On Crystalline Reflection and Refraction." *Trans. Roy. Irish Acad.* vols. XVIII. p. 31, and XXI. p. 17.

the plane of incidence, it is impossible to evade the conclusion, that the components perpendicular to the surface ought also to be continuous. In fact a discontinuity in the normal displacement, would involve something analogous to an area source in Hydrodynamics, and there are no grounds for supposing that anything of the kind occurs.

The condition that the normal displacements should be continuous, is

$$(A + A') \sin i = A_1 \sin r.$$

Multiplying this by (14) we obtain

$$(A^2 - A'^2) \sin i \cos i = A_1^2 \sin r \cos r \quad \text{.........(18)}.$$

From (7) the condition of continuity of energy may be written

$$(A^2 - A'^2)\, \rho \sin i \cos i = A_1^2\, \rho_1 \sin r \cos r,$$

and in order that this may be consistent with (18), we must have $\rho = \rho_1$. Accordingly Neumann and MacCullagh assumed this condition in their theories of reflection and refraction; and we shall now trace the consequences of this hypothesis.

When the vibrations are *perpendicular* to the plane of incidence, the equations are

$$A + A' = A_1,$$

$$(A^2 - A'^2) \sin 2i = A_1^2 \sin 2r,$$

whence

$$\left.\begin{aligned} A' &= \frac{A \tan (i - r)}{\tan (i + r)} \\ A_1 &= \frac{A \sin 2i}{\sin (i + r) \cos (i - r)} \end{aligned}\right\} \quad \text{..............(19)}.$$

When the vibrations are *in* the plane of incidence

$$\left.\begin{aligned} A' &= -\frac{A \sin (i - r)}{\sin (i + r)} \\ A_1 &= \frac{A \sin 2i}{\sin (i + r)} \end{aligned}\right\} \quad \text{..................(20)}.$$

It follows from (2), that on this theory, the intensity of light in all transparent media is proportional to the square of the amplitude, and accordingly (19) and (20) give the intensities of the reflected and refracted light. Neumann and MacCullagh further supposed, that the vibrations of polarized light are *in* instead of *perpendicular* to the plane of polarization, and on this supposition the formulæ (19) and (20) are in complete agreement with (16) and (11) given by the theory of Fresnel.

181. The two hypotheses of Neumann and MacCullagh are singularly seductive, inasmuch as it will hereafter be shown, that they enable the laws of the propagation of light in crystals, and also the reflection and refraction of light from crystalline surfaces, to be determined in accordance with Green's rigorous theory of elastic media; whereas the contrary assumption, that the density of the ether is different in different media, leads to a variety of difficulties in the application of this theory. There are however grave objections to these hypotheses; for in the first place, there are strong grounds for supposing, that the vibrations of polarized light are perpendicular to the plane of polarization; and in the second place, Lorenz and Lord Rayleigh have shown, as will be explained in Chapter XII., that the hypothesis of equal density, leads to the conclusion that there are two polarizing angles, which is contrary to experiment.

EXAMPLES.

1. A thin layer of fluid of thickness T, floats on the surface of a second fluid of infinitesimally greater refractive power. Light is incident perpendicularly on the layer; show that the intensity of the reflected light is

$$a^2\left(\frac{\mu-1}{\mu+1}\right)^2\left\{1+\frac{4\delta\mu}{\mu^2-1}\cos 4\pi T/\lambda\right\};$$

where μ and $\mu+\delta\mu$ are the refractive indices of the layer and of the fluid which supports it respectively, and a, λ are the amplitude and wave-length of the incident vibration.

2. If in the separating surface of two media, there be a straight groove of small depth c, inclined at an angle α to the plane of incidence, prove that there will be a groove in the refracted wave of depth $c \sin(i-r) \operatorname{cosec} i$, inclined at an angle $\tan^{-1}(\tan\alpha \operatorname{cosec} r)$ to the plane of refraction, where i and r are the angles of incidence and refraction.

What is the corresponding quantity for the reflected wave?

Explain why the image of a candle from rough glass becomes red, as the angle of incidence is diminished.

3. A ray polarized at right angles to the plane of incidence falls on a refracting surface; if the intensities of the reflected and refracted rays are equal, and the tangent of the polarizing angle lies between 1 and 3, prove that the corresponding angle of incidence is least, when the refracting medium is such that its polarizing angle is $\frac{1}{3}\pi$.

4. Circularly polarized light is incident in the usual manner upon a Fresnel's rhomb, so cut that after one reflection in the rhomb, the incident ray emerges from it perpendicularly to the cut face. A uniaxal crystal cut parallel to the optic axis is placed in the path of the emergent ray, with its faces normal to it, and with its principal plane inclined at an angle $\frac{1}{4}\pi$ to the plane of incidence and reflection in the rhomb. Show that the intensities of the refracted rays are as $\sqrt{2}-1 : \sqrt{2}+1$.

5. If light polarized perpendicularly to the plane of incidence, falls on a thin plate of air between two plates of different kinds of glass, prove that there are two angles at which the colours will disappear, and that between the two angles a change takes place in the order of the colours.

6. A ray of circularly polarized light is incident at the plane surface of separation of two media. If e and e' are the excentricities of the elliptically polarized light reflected and refracted, and i and r the angles of incidence and refraction, show that

$$(1-e^2)(1-e'^2) = \cos^2(i+r).$$

CHAPTER XI.

GREEN'S THEORY OF ISOTROPIC MEDIA.

182. THE various dynamical theories of the ether, which have been proposed to explain optical phenomena, may be classed under three heads; (i) theories which suppose that the ether possesses the properties of an elastic medium, which is capable of resisting compression and distortion; (ii) theories based upon the mutual reaction of ether and matter; (iii) the electromagnetic theory advanced by the late Prof. Clerk-Maxwell, which assumes that light is the result of an electromagnetic disturbance. We shall now proceed to consider the first class of theories.

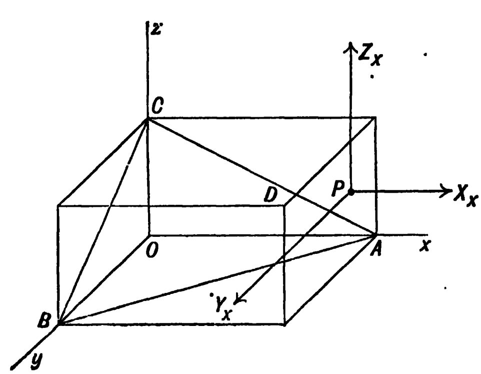
183. The dynamical theory proposed by Green[1], assumes that the ether is an elastic medium, which is capable of resisting compression and distortion. It therefore follows, that if the ether is in equilibrium, and any element is displaced from its position of equilibrium or is set in motion, the ether will be thrown into a state of strain, and will thereby acquire *potential energy*. Now the potential energy of any element of the ether, must necessarily depend upon the particular kind of displacement to which it is subjected; hence the potential energy per unit of volume must be a function of the displacements, or their differential coefficients, or of both. If therefore we can determine the form of this function, the equations of motion can be at once obtained by known dynamical methods.

According to the views held by Cauchy, the ether is to be regarded as a system of material particles acting upon one another by mutually attractive and repulsive forces, such that the

[1] *Trans. Camb. Phil. Soc.* 1838; and *Math. Papers*, p. 245.

mutual action between any two particles is along the line joining them; but inasmuch as the law of force is entirely a matter of speculation, Green discarded the hypothesis of mutually attracting particles, and based his theory upon the assumption that;—*In whatever way the elements of any material system may act upon one another, if all the internal forces be multiplied by the elements of their respective distances, the total sum for any assigned portion of the medium will be an exact differential of some function.* This function is what is now known as the potential energy of the portion of the medium considered; and Green showed that in its most general form, it is a homogeneous quadratic function of what, in the language of the Theory of Elasticity, are called the six components of strain, and therefore contains twenty-one terms, whose coefficients are constant quantities. For a medium which is symmetrical with respect to three rectangular planes, the expression for the potential energy involves nine independent constants; whilst for an isotropic medium it involves only two; one of which depends upon the resistance which the medium offers to compression, or change of volume unaccompanied by change of shape, whilst the other depends upon the resistance which the medium offers to distortion or shearing stress, unaccompanied by change of volume.

184. The general theory of media, which are capable of resisting compression and distortion, is given in treatises on Elasticity; and it will therefore be unnecessary to reproduce investigations, which are to be found in such works. There are however one or two points, which require consideration; and we shall commence by examining the stresses, which act upon an element of such a medium.



Let the figure represent a small parallelopiped of the medium. The stresses which act on the face AD are,

(i) A normal traction X_x parallel to Ox;

(ii) A tangential stress or shear Y_x parallel to Oy;

(iii) A tangential stress or shear Z_x parallel to Oz.

Similarly the stresses which act upon the faces BD and CD, are Y_y, Z_y, X_y and Z_z, X_z, Y_z.

These are the stresses exerted on the faces AD, BD, CD of the element by the surrounding medium; the stresses exerted by the medium on the three opposite faces will be in the opposite directions.

185. In order to find the equations of motion, let u, v, w be the displacements parallel to the axes, of any point x, y, z; ρ the density, and X, Y, Z the components of the impressed forces per unit of mass. Then resolving parallel to the axes, we obtain the equations

$$\left.\begin{aligned}\rho\frac{d^2u}{dt^2}&=\rho X+\frac{dX_x}{dx}+\frac{dX_y}{dy}+\frac{dX_z}{dz}\\ \rho\frac{d^2v}{dt^2}&=\rho Y+\frac{dY_x}{dx}+\frac{dY_y}{dy}+\frac{dY_z}{dz}\\ \rho\frac{d^2w}{dt^2}&=\rho Z+\frac{dZ_x}{dx}+\frac{dZ_y}{dy}+\frac{dZ_z}{dz}\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots(1).$$

These equations express the fact, that the rates of change of the components of the linear momentum of an element of the medium, are equal to the components of the forces which act upon the element. It is however also necessary, that the rates of change of the components of the angular momentum of the element about the axes, should be equal to the components of the couples which act upon the element. Whence taking moments about the axis of x, we obtain

$$\iiint\rho\left(y\frac{d^2w}{dt^2}-z\frac{d^2v}{dt^2}\right)dxdydz=\iiint\rho\,(yZ-zY)\,dxdydz$$

$$+\iint\{y\,(lZ_x+mZ_y+nZ_z)-z\,(lY_x+mY_y+nY_z)\}\,dS\ldots(2),$$

where dS is an element of the surface of the portion of the medium considered, and l, m, n are the direction cosines of the normal at dS.

Transforming the surface integral into a volume integral[1], and taking account of (1); (2) reduces to

$$\iiint (Z_y - Y_z)\, dx\,dy\,dz = 0 \quad\dots\dots\dots\dots\dots\dots (3),$$

which requires that $Z_y = Y_z$. It can similarly be shown, that $X_z = Z_x$, and $Y_x = X_y$. These results show, that the component stresses are completely specified by the *six* quantities X_x, Y_y, Z_z, Y_z, Z_x, X_y, which we shall denote by the letters P, Q, R, S, T, U.

Equations (1) may now be written

$$\left.\begin{aligned}\rho\frac{d^2u}{dt^2} &= \rho X + \frac{dP}{dx} + \frac{dU}{dy} + \frac{dT}{dz}\\ \rho\frac{d^2v}{dt^2} &= \rho Y + \frac{dU}{dx} + \frac{dQ}{dy} + \frac{dS}{dz}\\ \rho\frac{d^2w}{dt^2} &= \rho Z + \frac{dT}{dx} + \frac{dS}{dy} + \frac{dR}{dz}\end{aligned}\right\} \quad\dots(4).$$

186. It is important to notice, that the fundamental principles of Dynamics require that the relations

$$Z_y = Y_z, \quad X_z = Z_x, \quad Y_x = X_y \quad\dots\dots\dots\dots\dots(5),$$

should exist between these stresses, because certain theories have been proposed, involving assumptions respecting the mutual reaction of ether and matter, in which these conditions are not satisfied. If however the medium were supposed to possess gyrostatic momentum, the above conditions would not be fulfilled. The conception of a medium, possessing a distribution of gyrostats (or fly-wheels), by means of which it is endowed with angular momentum, is originally due to Sir W. Thomson[2], and suggests a means of explaining the rotatory effects of magnetism on light.

187. We are now prepared to apply the Theory of Elasticity to isotropic media, and we shall adopt the notation of Thomson and Tait's *Natural Philosophy*, for stresses, strains and elastic constants; so that k denotes the resistance to compression, n the rigidity, and we shall often employ m to denote $k + \frac{1}{3}n$.

[1] The transformation may be effected by the theorem proved in my treatise on *Hydrodynamics*, vol. I. § 7.

[2] *Proc. Lond. Math. Soc.* vol. VI. p. 190; Larmor, *Ibid.* vol. XXI. p. 423; and vol. XXIII.

The equations connecting the strains and displacements are

$$\left.\begin{aligned} e &= \frac{du}{dx}, & f &= \frac{dv}{dy}, & g &= \frac{dw}{dz} \\ a &= \frac{dw}{dy} + \frac{dv}{dz}, & b &= \frac{du}{dz} + \frac{dw}{dx}, & c &= \frac{dv}{dx} + \frac{du}{dy} \end{aligned}\right\} \quad \text{......(6).}$$

It is also convenient to denote the dilatation by δ, and the rotations by ξ, η, ζ, where these quantities are defined by the equations

$$\left.\begin{aligned} &\delta = e + f + g, \\ &\xi = \frac{dw}{dy} - \frac{dv}{dz}, \quad \eta = \frac{du}{dz} - \frac{dw}{dx}, \quad \zeta = \frac{dv}{dx} - \frac{du}{dy} \end{aligned}\right\} \quad \text{......(7).}$$

The rotations ξ, η, ζ are quantities analogous to the components of molecular rotation in Hydrodynamics.

The potential energy W of a homogeneous isotropic elastic medium is given in terms of the strains by the equation[1]

$$W = \tfrac{1}{2}(m+n)\delta^2 + \tfrac{1}{2}n\{a^2 + b^2 + c^2 - 4(ef + fg + ge)\} \text{...(8).}$$

The work done by the stresses P, Q, &c. in producing the infinitesimal strains δe, δf, &c. is

$$\delta W = P\delta e + Q\delta f + R\delta g + S\delta a + T\delta b + U\delta c \quad \text{......(9),}$$

whence

$$\left.\begin{aligned} \frac{dW}{de} &= P, \text{ \&c., \&c.} \\ \frac{dW}{da} &= S, \text{ \&c., \&c.} \end{aligned}\right\} \text{.....................(10),}$$

and therefore by (6) we obtain

$$\left.\begin{aligned} P &= (m-n)\delta + 2ne \\ Q &= (m-n)\delta + 2nf \\ R &= (m-n)\delta + 2ng \end{aligned}\right\} \text{..................(11),}$$

$$S = na, \quad T = nb, \quad U = nc.$$

[1] It is interesting to notice, that (8) is an example of an application of the Theory of Invariants to Physics. The three invariants, which involve the first and second powers of differential coefficients of the first order of u, v, w with respect to x, y, z are

$$e+f+g, \quad a^2+b^2+c^2-4(ef+fg+ge), \quad \text{and} \quad \xi^2+\eta^2+\zeta^2.$$

Since the value of W must be independent of the directions of the axes, and must also be a quadratic function, its most general form is

$$A\delta^2 + B\{a^2+b^2+c^2-4(ef+fg+ge)\} + C(\xi^2+\eta^2+\zeta^2).$$

Equations (5) require that $C=0$; whence the expression reduces to (8).

If the medium is isotropic, and no bodily forces act, the equations of motion are obtained by substituting the values of the stresses from (11) in (4), and taking account of (6); we thus obtain the following equations of motion in terms of the displacements, viz.

$$\left.\begin{aligned}\rho\frac{d^2u}{dt^2} &= m\frac{d\delta}{dx}+n\nabla^2u\\ \rho\frac{d^2v}{dt^2} &= m\frac{d\delta}{dy}+n\nabla^2v\\ \rho\frac{d^2w}{dt^2} &= m\frac{d\delta}{dz}+n\nabla^2w\end{aligned}\right\}\quad\text{.(12).}$$

These are the equations of motion of the ether according to Green's Theory.

188. If we differentiate (12) with respect to x, y, z, add and take account of (7), we obtain

$$\rho\frac{d^2\delta}{dt^2}=(m+n)\nabla^2\delta\ldots\ldots\ldots\ldots\ldots\ldots\text{(13).}$$

If we eliminate δ between each pair of equations (12) and take account of (7), we obtain

$$\left.\begin{aligned}\rho\frac{d^2\xi}{dt^2} &= n\nabla^2\xi\\ \rho\frac{d^2\eta}{dt^2} &= n\nabla^2\eta\\ \rho\frac{d^2\zeta}{dt^2} &= n\nabla^2\zeta\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(14).}$$

It therefore follows, that the waves which can be propagated in the medium consist of two distinct types; one of which involves change of volume without rotation, and whose velocity of propagation is seen from (13) to be equal to $(m+n)^{\frac{1}{2}}/\rho^{\frac{1}{2}}$; whilst the other involves rotation and distortion, without change of volume, and whose velocity of propagation is $(n/\rho)^{\frac{1}{2}}$. The first type of waves depends partly on the rigidity and partly on the elasticity of volume; whilst the second type depends solely on the rigidity, and is therefore incapable of being propagated in a medium devoid of rigidity, such as a perfect gas. Hence if any disturbance, which involves change of volume and distortion, be communicated to a portion of the medium, two distinct trains of waves will be produced; one of which consists of a condensation

and rarefaction, which is propagated with a velocity $(m+n)^{\frac{1}{2}}/\rho^{\frac{1}{2}}$; whilst the other consists of a distortion or change of shape, which is propagated with a velocity $(n/\rho)^{\frac{1}{2}}$.

189. Let us now suppose that a train of plane waves is propagated through the medium, the direction cosines of whose fronts are l, m, n. Since equations of the form (13) and (14) are satisfied by the function

$$F(lx+my+nz-Vt),$$

where V is the velocity of propagation, we may suppose that the resultant displacement S is

$$S=F(lx+my+nz-Vt);$$

hence if λ, μ, ν be the direction cosines of the direction of displacement, we shall have

$$u=S\lambda, \quad v=S\mu, \quad w=S\nu.$$

Whence
$$\delta=(l\lambda+m\mu+n\nu)S',$$
$$\xi=(m\nu-n\mu)S',$$
$$\eta=(n\lambda-l\nu)S',$$
$$\zeta=(l\mu-m\lambda)S'.$$

If the direction of displacement is in the front of the wave, so that the vibrations are transversal,

$$l\lambda+m\mu+n\nu=0,$$

whence
$$\delta=0.$$

If on the other hand the displacement is perpendicular to the front of the wave, so that the vibrations are longitudinal,

$$l=\lambda, \; m=\mu, \; n=\nu$$

whence
$$\xi=0, \; \eta=0, \; \zeta=0, \; \delta=S'.$$

It therefore follows that when the vibrations are longitudinal, dilatational waves unaccompanied by rotation or distortion are alone propagated; whilst if the vibrations are transversal, distortional waves unaccompanied by condensation or rarefaction are alone propagated.

Since the phenomenon of polarization compels us to adopt the hypothesis, that the vibrations which constitute light are transversal and not longitudinal, we must suppose that the portion of the disturbance, which consists of distortional vibrations, is alone capable of affecting the eye.

190. Let us now suppose, that a wave of light consisting of transversal vibrations *in* the plane of incidence, is refracted through a prism. Then it is not difficult to show, that the incident wave will give rise to two refracted waves, which respectively consist of transversal and longitudinal vibrations; and since the velocities of propagation of these two waves are different, their indices of refraction will be different, and thus the two refracted rays will not coincide. The refracted wave, whose vibrations are normal to the wave-front, will be divided on emergence at the second face of the prism into two more refracted waves, one of which will consist of transversal and the other of normal vibrations. Thus, even though we assumed that waves consisting of normal vibrations are incapable of affecting the eye; it would follow, in the first place, that a wave of normal vibrations *might* give rise to a wave of transversal vibrations, and consequently the sensation of light might be produced by something which is not light; and in the second place, that whenever light polarized perpendicularly to the plane of incidence is refracted through a prism, there ought to be two refracted rays. This is altogether contrary to experience, hence our theory of the ether without some further modification is defective. When we discuss the reflection and refraction of light, it will be proved that the refracted wave whose vibrations are normal, will become insensible in the second medium, at a distance from the face of the prism which is equal to a few wave-lengths, *provided the ratio of the velocity of propagation of the normal wave to that of the transversal wave, is either very great or very small;* that is whenever $(m+n)/n$ is very large or very small. Now $(m+n)/n = (k+\frac{4}{3}n)/n$, hence this ratio will be large, if k is large compared with n. But if a uniform hydrostatic pressure be applied to every point of the surface of a spherical portion of the medium, k is the ratio of the pressure to the compression produced. If therefore k is large compared with n, the power which the medium possesses of resisting compression, must be very great in comparison with its power of resisting distortion. On the other hand, if the ratio $(k+\frac{4}{3}n)/n$ were positive and very small, it would be necessary for k to be a negative quantity, whose numerical value is equal to or slightly less than $\frac{4}{3}n$. But inasmuch as in this case, an increase of pressure would produce an increase of volume, and as no known substance possesses this property, Green concluded that k is very large compared with n. The difficulty of satisfactorily accounting

for waves of normal vibrations, whose existence we are forced to admit, is thus to a great extent, though not entirely, overcome. For since the velocity of light in vacuo is about 299,860 kilometres per second, the velocity of propagation of the condensational waves would be enormously greater than those of the distortional waves; and it is therefore not unreasonable to suppose, that they are incapable of affecting our eyes. At the same time, the amount of energy existing in the universe, which is due to these waves, must be very large; and the assumption, that this large amount of energy is incapable of producing any effect of which our senses are capable of taking cognizance, is not very satisfactory.

Sir W. Thomson has lately proposed a theory, in which it is assumed that k is a negative quantity, which is numerically equal to, or slightly less than $\frac{4}{3}n$. This theory will be considered when we discuss double refraction; for the present we shall confine our attention to Green's theory, in which k is supposed to be large compared with n.

191. Green's hypothesis has sometimes been supposed to involve the assumption, that the ether is very nearly incompressible. This however is an altogether erroneous view, for as a matter of fact the ether might be more compressible than the most highly compressible gas; all that is necessary is, that the *ratio* of the resistance to compression to the resistance to distortion should be very large. Moreover since the velocity of light in vacuo is about 299,860 kilometres per second, ρ must be very small in comparison with n. That k, n and ρ must all be exceedingly small quantities compared with ordinary standards, is proved from the fact, that the most delicate astronomical observations have not succeeded in detecting with any certainty, that any resistance is offered by the ether to the motions of the planets; although it has been suggested that the irregularities, which are observed in the motions of some of the comets, may be referred to this cause. But whether this be so or not, Green's hypothesis requires us to suppose, that ρ, n and k are exceedingly small quantities in ascending order of magnitude, such that n is large in comparison with ρ, and k is large compared with n.

CHAPTER XII.

APPLICATIONS OF GREEN'S THEORY.

192. In the present chapter, we shall apply Green's theory to investigate the reflection and refraction of light at the surface of two isotropic media, the theory of Newton's rings, and the reflection of light from a pile of plates.

We have stated in § 10, that the intensity of light is measured by the mean energy per unit of volume. Hence if T and W denote the kinetic and potential energies of a plane wave per unit of volume, which is being propagated parallel to x, we have

$$T = \tfrac{1}{2}\rho\dot{w}^2, \quad W = \tfrac{1}{2}n\left(\frac{dw}{dx}\right)^2.$$

Hence if $$w = A\cos\frac{2\pi}{\lambda}(x - Vt),$$

then $$T = \frac{2\pi^2 V^2 A^2 \rho}{\lambda^2}\sin^2\frac{2\pi}{\lambda}(x - Vt)$$

$$W = \frac{2\pi^2 A^2 n}{\lambda^2}\sin^2\frac{2\pi}{\lambda}(x - Vt);$$

and since $n = V^2\rho$, it follows that the kinetic and potential energies are equal. Accordingly the mean energy per unit of volume is

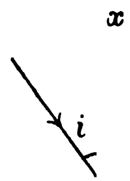
$$E = 2\pi^2 A^2 n/\lambda^2 = 2\pi^2 A^2 n/V^2\tau^2 \qquad (1)$$

which measures the intensity.

Reflection and Refraction[1].

193. We are now prepared to consider the problem of reflection and refraction.

Let the axis of x be perpendicular to the reflecting surface, and let the axis of z be parallel to the intersections of the wave-fronts with the same surface; and let us first suppose, that the incident light is polarized in the plane of incidence.



Since u and v are zero, it follows from (12) of § 187, that the equations of motion are

$$\frac{d^2w}{dt^2} = V^2\left(\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2}\right) \quad\text{.....................(2)}$$

in the first medium and

$$\frac{d^2w_1}{dt^2} = V_1^2\left(\frac{d^2w_1}{dx^2} + \frac{d^2w_1}{dy^2}\right) \quad\text{..................(3)}$$

in the second, where $V^2 = n/\rho$, $V_1^2 = n_1/\rho_1$.

The conditions to be satisfied at the surface of separation are, that the displacements and stresses must be the same in both media. These conditions will often be referred to, as the conditions of continuity of displacement and stress.

The first condition gives

$$w = w_1 \quad\text{..............................(4)},$$

and the second

$$n\frac{dw}{dx} = n_1\frac{dw_1}{dx} \quad\text{.......................(5)}.$$

[1] Green, *Math. Papers*, pp. 245, 283. Hon. J. W. Strutt (Lord Rayleigh), *Phil. Mag.* Aug. 1871. Kurz, *Pogg. Ann.* vol. CVIII. p. 396.

Let A, A', A_1 be the amplitudes of the incident, reflected and refracted waves, then we may write

$$w = A\epsilon^{\iota\kappa(lx+my-Vt)} + A'\epsilon^{\iota\kappa(l'x+m'y-Vt)} \quad\text{...........(6)},$$

$$w_1 = A_1\epsilon^{\iota\kappa_1(l_1x+m_1y-V_1t)} \quad\text{...........................(7)},$$

where $$\kappa = 2\pi/\lambda, \quad \kappa_1 = 2\pi/\lambda_1.$$

The vibration in the second medium is a forced vibration produced and maintained by the incident waves, hence the coefficients of t must be the same in all three waves. Also the coefficients of y must be the same, since the traces of all three waves on the plane $x=0$ move together. Hence

$$\kappa V = \kappa_1 V_1, \quad \kappa m = \kappa_1 m_1 \quad\text{....................(8)}.$$

But $$\left.\begin{array}{lll} l = -\cos i, & l' = \cos i, & l_1 = -\cos r \\ m = \sin i, & m' = \sin i, & m_1 = \quad\sin r \end{array}\right\} \quad\text{.........(9)}.$$

From these equations we see that

$$\frac{V}{V_1} = \frac{\sin i}{\sin r} = \frac{\lambda}{\lambda_1} \quad\text{.......................(10)}.$$

The first of these equations is the well-known law of sines, whilst the second expresses the condition, that the period of the refracted wave must be equal to that of the incident.

Substituting from (6) and (7) in (4) and (5), we obtain

$$A + A' = A_1,$$

$$(A - A')\, n\kappa \cos i = A_1 n_1 \kappa_1 \cos r,$$

the last of which may be written

$$A - A' = \frac{A_1 n_1 \tan i}{n \tan r};$$

whence $$\left.\begin{array}{l} A' = -\dfrac{A\,(n_1 \tan i - n \tan r)}{n_1 \tan i + n \tan r} \\[2ex] A_1 = \dfrac{2An \tan r}{n_1 \tan i + n \tan r} \end{array}\right\} \quad\text{..............(11)}.$$

If I, I', I_1 be the square roots of the intensities, it follows from (1) that

$$\frac{I}{An^{\frac{1}{2}}/V} = \frac{I'}{A'n^{\frac{1}{2}}/V} = \frac{I_1}{A_1 n_1^{\frac{1}{2}}/V_1} \quad\text{...............(12)}.$$

Up to the present time, we have not assumed that any relation exists between n and n_1. If however we assume with Green, that the rigidities are the same in all isotropic media, and

that refraction is consequently due to a difference of density, we must put $n = n_1$ and we obtain

$$I' = -\frac{I \sin(i - r)}{\sin(i + r)} \quad\text{.................... (13)},$$

$$I_1 = \frac{I \sin 2i}{\sin(i + r)} \quad\text{.......................... (14)},$$

which are the same as Fresnel's formulæ.

If on the other hand we adopt the hypothesis of Neumann and MacCullagh, that the density of the ether is the same in all media, and that refraction is consequently due to a difference of rigidity, it follows that

$$\frac{n}{n_1} = \frac{V^2}{V_1^2} = \frac{\sin^2 i}{\sin^2 r},$$

and we obtain

$$I' = \frac{I \tan(i - r)}{\tan(i + r)} \quad\text{....................... (15)},$$

$$I_1 = \frac{I \sin 2i}{\sin(i + r)\cos(i - r)} \quad\text{.............. (16)}.$$

These expressions are the same as Fresnel's formulæ for the intensity of light polarized perpendicularly to the plane of incidence.

194. The preceding formulæ do not enable us to decide the question, whether the vibrations of polarized light are in or perpendicular to the plane of polarization; or whether the hypothesis of Green on the one hand, or of Neumann and MacCullagh on the other, is the best representative of the facts. For the present we shall adopt Green's view, and shall proceed to calculate the change of phase which occurs, when the angle of incidence exceeds the critical angle.

If μ is the index of refraction

$$\mu = \frac{\sin i}{\sin r} = \frac{V}{V_1} = \sqrt{\frac{\rho_1}{\rho}};$$

hence if $\rho_1 > \rho$, $\mu > 1$, and r is always real; but if $\rho_1 < \rho$, $\mu < 1$, and the angle of refraction becomes imaginary, when the angle of incidence exceeds the critical angle.

When $\rho_1 < \rho$, we shall write μ^{-1} for μ, so that μ denotes the index of refraction from the rarer into the denser medium, and

$$\mu \sin i = \sin r.$$

Since the expressions for the amplitudes of the reflected and refracted waves become complex, we must write

$$w = A\epsilon^{\iota\kappa(lx+my-Vt)} + (\alpha+\iota\beta)\,\epsilon^{\iota\kappa(-lx+my-Vt)} \quad\text{.........(17)},$$

$$w_1 = (\alpha_1+\iota\beta_1)\,\epsilon^{\iota\kappa_1(l_1x+m_1y-V_1t)} \quad\text{.........................(18)},$$

in which α, β, α_1, β_1, are real. Now

$$l_1 = -\cos r = \pm\,\iota\,(\mu^2\sin^2 i - 1)^{\frac{1}{2}}$$

$$\kappa_1 = \kappa/\mu,$$

whence $\quad \iota\kappa_1 l_1 = (2\pi/\lambda\mu)(\mu^2\sin^2 i - 1)^{\frac{1}{2}} = \kappa a_1$ (say),

the lower sign being taken, because x is negative in the second medium.

The boundary conditions give

$$A + \alpha + \iota\beta = \alpha_1 + \iota\beta_1$$

$$(A - \alpha - \iota\beta)\cos i = (\alpha_1 + \iota\beta_1)\,\iota a_1.$$

Equating the real and imaginary parts, we obtain

$$\alpha = \frac{A(\cos^2 i - a_1^2)}{\cos^2 i + a_1^2},\quad \alpha_1 = \frac{2A\cos^2 i}{\cos^2 i + a_1^2},$$

$$\beta = \beta_1 = -\frac{2Aa_1\cos i}{\cos^2 i + a_1^2}.$$

Let $\quad \tan\dfrac{\pi e}{\lambda} = \dfrac{a_1}{\cos i} = \dfrac{(\mu^2\sin^2 i - 1)^{\frac{1}{2}}}{\mu\cos i}$ ....(19),

then $\quad \alpha + \iota\beta = A\epsilon^{-2\iota\pi e/\lambda},\ \alpha_1 + \iota\beta_1 = 2A\epsilon^{-\iota\pi e/\lambda}\cos\pi e/\lambda$;

whence the reflected wave is

$$w = A\cos\frac{2\pi}{\lambda}(x\cos i + y\sin i - Vt - e)$$

and the refracted wave is

$$w_1 = \frac{2A\mu\cos i}{(\mu^2-1)^{\frac{1}{2}}}\,\epsilon^{\frac{2\pi}{\lambda\mu}(\mu^2\sin^2 i - 1)^{\frac{1}{2}}x}\cos\frac{2\pi}{\lambda}(y\sin i - Vt - \tfrac{1}{2}e)\text{...(20)}.$$

From these equations we see, that when the angle of incidence exceeds the critical angle, the reflection is total and is accompanied by a change of phase, whose value is given by (19).

Since the refracted wave involves an exponential term in the amplitude, it becomes insensible at a distance from the surface which is equal to a few wave-lengths.

All the foregoing results are in agreement with Fresnel's formulæ.

195. The investigation of the problem, when the light is polarized perpendicularly to the plane of incidence is more difficult.

In this case $w=0$, and by (12) of § 187, the equations of motion in the upper medium are

$$\left.\begin{aligned} \rho\frac{d^2u}{dt^2} &= (k+\tfrac{1}{3}n)\frac{d}{dx}\left(\frac{du}{dx}+\frac{dv}{dy}\right)+n\left(\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}\right) \\ \rho\frac{d^2v}{dt^2} &= (k+\tfrac{1}{3}n)\frac{d}{dy}\left(\frac{du}{dx}+\frac{dv}{dy}\right)+n\left(\frac{d^2v}{dx^2}+\frac{d^2v}{dy^2}\right) \end{aligned}\right\} \quad \text{..(21)},$$

with similar equations for the lower medium. In these equations k is the resistance to compression, which Green supposes to be very large compared with n.

The boundary conditions are

$$u=u_1, \quad v=v_1 \quad \text{.......................(22)},$$

$$\left.\begin{aligned} \left(k+\frac{4}{3}n\right)\frac{du}{dx}+\left(k-\frac{2}{3}n\right)\frac{dv}{dy} &= \left(k_1+\frac{4}{3}n_1\right)\frac{du_1}{dx}+\left(k_1-\frac{2}{3}n_1\right)\frac{dv_1}{dy} \\ n\left(\frac{du}{dy}+\frac{dv}{dx}\right) &= n_1\left(\frac{du_1}{dy}+\frac{dv_1}{dx}\right) \end{aligned}\right\} \text{..(23)},$$

of which the first two express the conditions of continuity of displacement, and the last two the conditions of continuity of stress.

We have therefore *four* equations to determine *two* unknown quantities. Now we have already shown, that elastic media are capable of propagating waves of two distinct types; viz. dilatational waves, which involve condensation and rarefaction, and distortional waves, which involve change of shape without change of volume. When the vibrations of the incident wave are not parallel to the reflecting surface, there will be a *dilatational* as well as a distortional reflected and refracted wave, whose amplitudes must be determined by (22) and (23); accordingly we have four unknown quantities and four equations to determine them. When the resistance to compression is very large, δ or $du/dx + dv/dy$ is very small, but we are not at liberty to treat the latter quantity as zero, because $k\delta$ is finite; we must therefore introduce the dilatational waves.

Let $$u=\frac{d\phi}{dx}+\frac{d\psi}{dy}, \quad v=\frac{d\phi}{dy}-\frac{d\psi}{dx} \quad \text{..............(24)},$$

$$U^2=(k+\tfrac{4}{3}n)/\rho, \quad V^2=n/\rho \quad \text{.................... (25)},$$

so that U and V are the velocities of the dilatational and distortional waves respectively. Then

$$\frac{du}{dx}+\frac{dv}{dy}=\nabla^2\phi,\quad \frac{du}{dy}-\frac{dv}{dx}=\nabla^2\psi,$$

where, by (21), ϕ and ψ respectively satisfy the equations

$$\left.\begin{aligned}\frac{d^2\phi}{dt^2}&=U^2\nabla^2\phi\\ \frac{d^2\psi}{dt^2}&=V^2\nabla^2\psi\end{aligned}\right\}\quad\text{.................. (26)},$$

with similar equations for the lower medium.

Let us now assume that in the first medium

$$\left.\begin{aligned}\psi&=A\epsilon^{\iota\kappa(lx+my-Vt)}+A'\epsilon^{\iota\kappa(-lx+my-Vt)}\\ \phi&=B'\epsilon^{\iota\kappa(ax+my-Vt)},\end{aligned}\right\}\quad\text{...... (27)},$$

and in the second medium

$$\left.\begin{aligned}\psi_1&=A_1\epsilon^{\iota\kappa_1(l_1x+m_1y-V_1t)}\\ \phi_1&=B_1\epsilon^{\iota\kappa_1(a_1x+m_1y-V_1t)}\end{aligned}\right\}\quad\text{.............. (28)}.$$

In these equations the coefficients of y and t must be the same in all the waves, whence

$$\kappa m=\kappa_1 m_1,\quad \kappa V=\kappa_1 V_1 \text{ .... (29)},$$

also since the dilatational wave is propagated with velocity U, it follows that its wave-length is equal to $U\lambda/V$ which is therefore very large compared with λ.

Substituting from the second of (27) in the first of (26), we obtain

$$V^2=(a^2+m^2)U^2 \text{ .... (30)},$$

and since V/U is very small, we shall have to a sufficient approximation

$$\iota a=-m;$$

the lower sign being taken, because x is positive in the upper medium.

Similarly from the second of (28) and (26), we shall obtain

$$\kappa_1^2 V_1^2=\kappa_1^2(a_1+m_1^2)\,U_1^2=(\kappa_1^2a_1^2+\kappa^2m^2)\,U_1^2 \text{ ...... (31)}.$$

Whence approximately

$$\iota\kappa_1 a_1=\kappa m;$$

because x is negative in the lower medium.

We may therefore write

$$\left.\begin{aligned}\phi &= B'\epsilon^{-\kappa m x+\iota\kappa(my-Vt)}\\ \phi_1 &= B_1\epsilon^{\kappa m x+\iota\kappa(my-Vt)}\end{aligned}\right\}\quad\text{.................}(32).$$

From (22) and (24) we obtain

$$\left.\begin{aligned}-B'+\iota(A+A') &= B_1+\iota A_1\\ B'm-(A-A')\,l &= B_1 m - A_1\kappa_1 l_1/\kappa\end{aligned}\right\}\quad\text{..........}(33).$$

We shall now assume with Green, that $n = n_1$; whence, since the continuity of u involves the continuity of du/dy, the last of (23) reduces to

$$dv/dx = dv_1/dx,$$

or

$$-B'm^2 - (A + A')\,\iota l^2 = B_1 m^2 - A_1\iota\kappa_1{}^2 l_1{}^2/\kappa^2\text{ ....}(34).$$

The first of (23) may be written

$$\rho U^2\nabla^2\phi - 2n\frac{dv}{dy} = \rho_1 U_1{}^2\nabla^2\phi_1 - 2n\frac{dv_1}{dy},$$

which, through the continuity of v and dv/dy, becomes

$$\rho U^2\nabla^2\phi = \rho_1 U_1{}^2\nabla^2\phi_1.$$

Substituting in this from the second of (27) and (28), we obtain

$$B'\rho U^2\kappa^2(a^2+m^2) = B_1\rho_1 U_1{}^2\kappa_1{}^2(a_1{}^2+m_1{}^2);$$

or

$$B'/\lambda^2 = B_1/\lambda_1{}^2\text{ ....}(35)$$

by (30) and (31); since

$$V^2\rho = V_1{}^2\rho_1 = n,\ \text{ and }\ \kappa = 2\pi/\lambda.$$

Introducing the index of refraction μ, which is equal to λ/λ_1, (35) may be written

$$B'/B_1 = \mu^2.$$

From this equation, (34), and the last of (33), we obtain

$$B' = \frac{\iota A_1\mu^2(\mu^2-1)}{\mu^2+1},\qquad B_1 = \frac{\iota A_1(\mu^2-1)}{\mu^2+1}\text{ ....}(36).$$

Using these in (33), we obtain

$$A + A' = A_1\mu^2,$$

$$A - A' = A_1\left\{\frac{\tan i}{\tan r} - \frac{\iota(\mu^2-1)^2}{\mu^2+1}\tan i\right.$$

and therefore

$$\left.\begin{aligned}2A &= A_1\left\{\mu^2 + \frac{\tan i}{\tan r} - \frac{\iota(\mu^2-1)^2}{\mu^2+1}\tan i\right\}\\ 2A' &= A_1\left\{\mu^2 - \frac{\tan i}{\tan r} + \frac{\iota(\mu^2+1)^2}{\mu^2+1}\tan i\right\}\end{aligned}\right\}\quad\text{........}(37).$$

In order to realize these formulæ, let

$$2A = A_1 R\epsilon^{\iota e}, \qquad 2A' = A_1 R' \epsilon^{-\iota e'};$$

then
$$A'/A = (R'/R)\,\epsilon^{-\iota(e'+e)}, \qquad A_1 = 2A/R\epsilon^{\iota e};$$

whence if
$$M = (\mu^2 - 1)/(\mu^2 + 1),$$

$$\tan e = \frac{M(\mu^2 - 1)\tan i \tan r}{\mu^2 \tan r + \tan i}$$

$$= M \tan(i - r) \quad\text{..................(38)},$$

since
$$\mu = \sin i/\sin r.$$

Similarly
$$\tan e' = -M\tan(i + r) \quad\text{..................(39)},$$

whence

$$\left.\begin{aligned} \psi' &= (AR'/R)\cos\{\kappa(-lx + my - Vt) - e - e'\} \\ \psi_1 &= 2AR^{-1}\cos\{\kappa_1(l_1 x + m_1 y - V_1 t) - e\} \end{aligned}\right\}\text{.....(40)}.$$

These expressions show, that when light is polarized perpendicularly to the plane of incidence, the reflected and refracted waves experience a change of phase, which is determined by (38) and (39).

The amplitudes of the reflected and refracted waves are determined by the equations

$$\frac{R'^2}{R^2} = \frac{(\mu^2\cot i - \cot r)^2 + M^2(\mu^2 - 1)^2}{(\mu^2\cot i + \cot r)^2 + M^2(\mu^2 - 1)^2}$$

$$= \frac{\cot^2(i + r) + M^2}{\cot^2(i - r) + M^2} \quad\text{.......................(41)},$$

and

$$\frac{4}{R^2} = \frac{4\sin^2 r\cos^2 i}{\sin^2 i \sin^2(i + r)\{\cos^2(i - r) + M\sin^2(i - r)\operatorname{cosec}^2 r\}} \quad\text{...(42)}.$$

196. These equations do not agree with the results furnished by Fresnel's theory unless $M = 0$.

According to Fresnel's formula, when light polarized perpendicularly to the plane of incidence is incident at the polarizing angle, the intensity of the reflected light is zero; experiment however shows that this result is not rigorously correct, inasmuch as the intensity of the reflected light does not absolutely vanish, but attains a minimum value. On the other hand Green's formula deviates too much the other way, and shows that too much light is reflected at the polarizing angle.

197. To calculate the change of phase, when light is reflected at the surface of a rarer medium at an angle greater than the critical angle, we shall denote as before the index of refraction from the rarer to the denser medium by μ; we must therefore write μ^{-1} for μ in (37).

Now, $\quad \sin r = \mu \sin i,$

$$\tan r = \pm \frac{\mu \sin i}{\iota (\mu^2 \sin^2 i - 1)^{\frac{1}{2}}},$$

and
$$\frac{\kappa_1 l_1}{\kappa l} = \frac{\tan i}{\tan r} = \pm \frac{\iota (\mu^2 \sin^2 i - 1)^{\frac{1}{2}}}{\mu \cos i};$$

also since $\iota\kappa_1 l_1$ is a real positive quantity, and $l = -\cos i$, the upper sign must be taken; accordingly (37) become

$$2A = A_1 \left\{\frac{1}{\mu^2} + \frac{\iota (\mu^2 \sin^2 i - 1)^{\frac{1}{2}}}{\mu \cos i} - \frac{\iota (\mu^2 - 1)}{\mu^2} M \tan i\right\},$$

$$2A' = A_1 \left\{\frac{1}{\mu^2} - \frac{\iota (\mu^2 \sin^2 i - 1)^{\frac{1}{2}}}{\mu \cos i} + \frac{\iota (\mu^2 - 1)}{\mu^2} M \tan i\right\}.$$

These equations may be written in the form

$$2A = A_1 R \epsilon^{\iota\pi e/\lambda}, \quad 2A' = A_1 R \epsilon^{-\iota\pi e/\lambda};$$

from which we get $\quad A' = A \epsilon^{-2\iota\pi e/\lambda},$

where
$$\tan \frac{\pi e}{\lambda} = \mu (\mu^2 \tan^2 i - \sec^2 i)^{\frac{1}{2}} - \frac{(\mu^2 - 1)^2}{\mu^2 + 1} \tan i \text{ ....(43).}$$

Accordingly the reflected wave is

$$\psi' = A \cos \frac{2\pi}{\lambda} (-lx + my - Vt - e),$$

which shows that the reflection is total, and is accompanied by a change of phase, whose value is given by (43). The first term of the change of phase agrees with that which we have already obtained from Fresnel's theory, whilst the second is small unless the medium is highly refracting, or the angle of incidence is large.

When the incident light is plane polarized in any azimuth, the incident vibrations may be resolved into two components, which are respectively in and perpendicular to the plane of incidence; and from (19) and (43), we see that when the light is totally reflected, total reflection is accompanied by a change of phase, whose value is *not the same* for the two components.

Accordingly the reflected light under these circumstances is *elliptically polarized.* This remark will be found to be of importance, when we consider the selective reflection, which is produced by certain of the aniline dyes.

198. We have already pointed out, that in the theories of Neumann and MacCullagh, it is assumed that $\rho=\rho_1$, in which case reflection and refraction would be due to a difference of rigidity. We have also seen, that in Fresnel's theory the conditions of continuity of displacement violate the condition of continuity of energy, unless $\rho=\rho_1$. It therefore becomes important to enquire what results would be furnished by Green's theory, on the supposition that the density of the ether is the same in all media. This point has been examined by Lorenz[1] and Lord Rayleigh[2], and the results are decisive against the hypothesis in question.

In this case, the boundary conditions (33), which express continuity of displacement, remain the same as before; whilst (23) become

$$U^2\left(\frac{du}{dx}+\frac{dv}{dy}\right)-2V^2\frac{dv}{dy}=U_1^2\left(\frac{du_1}{dx}+\frac{dv_1}{dy}\right)-2V_1^2\frac{dv_1}{dy},$$

$$V^2\left(\frac{du}{dy}+\frac{dv}{dx}\right)=V_1^2\left(\frac{du_1}{dy}+\frac{dv_1}{dx}\right).$$

From the first equation, we obtain

$$B'U^2\kappa^2(a^2+m^2)-2V^2\kappa^2m\{B'm-(A-A')l\}$$
$$=B_1U_1^2\kappa_1^2(a_1^2+m_1^2)-2V_1^2\kappa_1^2m_1(B_1m_1-A_1l_1),$$

which by (29), (30) and (31) becomes

$$B'(1-2m^2)+2(A-A')lm=B_1(1-2m_1^2)+2A_1l_1m_1\ \ldots(44).$$

The second equation becomes

$$2B'am-(A+A')(l^2-m^2)=2B_1a_1m_1-A_1(l_1^2-m_1^2),$$

which reduces to

$$2B'm^2+\iota(A+A')(l^2-m^2)=-2B_1m_1^2+\iota A_1(l_1^2-m_1^2)\ldots(45),$$

since $$a=\iota m,\quad a_1=-\iota m_1.$$

From (33), we get

$$2B'=(A+A')\iota+(A-A')l/m-A_1(\mu l_1/m+\iota),$$
$$2B_1=(A+A')\iota-(A-A')l/m+A_1(\mu l_1/m-\iota).$$

[1] *Pogg. Ann.*, vol. CXIV.

[2] Hon. J. W. Strutt, *Phil. Mag.*, Aug. 1871.

Substituting the values of B', B_1 in (44) and (45) and reducing, we obtain

$$(A-A')\cot i - A_1 \cot r =$$
$$-\frac{(\mu^2-1)\sin^2 i}{\mu^2}\{(A+A'-A_1)\iota + (A-A')\cot i + A_1\cot r\}\ldots(46),$$

and

$$(A+A')(1+\mu^2\cot^2 i) - A_1(\mu^2+\cot^2 r) =$$
$$-\iota(\mu^2-1)\{(A-A')\cot i - A_1\cot r\}\ldots\ldots(47).$$

It will be sufficient for our purpose to calculate the intensity of the reflected light, when the difference between the refrangibilities of the two media is small. In this case, the intensity A' of the reflected light is small, and A_1 and r are nearly equal to A and i respectively. We may also neglect squares and higher powers of μ^2-1, hence in the small terms we may put $A'=0$, $r=i$, $A=A_1$; we thus obtain from (46) and (47)

$$A - A' = A_1\left\{\frac{\cot r}{\cot i} - \frac{2(\mu^2-1)\sin^2 i}{\mu^2}\right\},$$

$$A + A' = \frac{A_1(\mu^2+\cot^2 r)}{\mu^2\cot^2 i + 1}.$$

Now $$\cot^2 r = \frac{\mu^2-1}{\sin^2 i} + \cot^2 i,$$

therefore $$\cot r = \cot i\left(1 + \frac{\mu^2-1}{2\cos^2 i}\right),$$

approximately; hence

$$A - A' = A_1\{1 + (\mu^2-1)(\tfrac{1}{2}\sec^2 i - 2\sin^2 i)\},$$

and $$A + A' = A_1\{1 + 2(\mu^2-1)\sin^2 i\};$$

accordingly $$A' = -\tfrac{1}{4}A(\mu^2-1)\cos 4i\sec^2 i.$$

From this expression, we see that A' vanishes when $\cos 4i = 0$, that is when $i=\frac{1}{8}\pi$, $\frac{3}{8}\pi$. Under these circumstances, there would be two polarizing angles; and since nothing of the kind has ever been observed, we conclude that the hypothesis that $\rho=\rho_1$ is untrue, and that the rigidity of the ether in all media is the same.

On Newton's Rings.

199. In the investigation of Newton's rings, which is given in Chapter III., the assumption is tacitly made, that the angle of incidence is less than the critical angle; but when these rings are formed between the under surface of a prism and the upper surface of a lens, there is no difficulty in increasing the angle of incidence on the under surface of the prism, until it exceeds the critical angle. Under these circumstances it is found, that as soon as the critical angle is passed, the rings disappear, but the central black spot remains. Now the theories of Green and Fresnel, and also the electromagnetic theory, as we shall see later on, show that when the angle of incidence exceeds the critical angle, a superficial wave exists in the second medium, which penetrates a few wave-lengths, and then disappears. In the experiment of Newton's rings, the stratum of air between the prism and the lens is so exceedingly thin, as to be comparable with the wave-length of light; accordingly, the existence of the central spot beyond the critical angle was attributed by Dr Lloyd[1] to the disturbance in the second medium, which takes the place of that which, when the angle of incidence is less than the critical angle, constitutes the refracted light. We shall therefore proceed to give an investigation due to Stokes[2], in which the superficial wave is taken into account.

200. Adopting the notation for the colours of thin plates, let the amplitude of the incident vibration in the glass be taken as unity, let b, c be the amplitudes of the reflected and refracted vibrations, when light passes from glass to air, and let e and f be the corresponding quantities, when light passes from air to glass. Also let us first suppose, that the stratum of air is bounded by two parallel planes at a distance D.

Taking the origin in the surface of the upper plate of glass, let the incident vibration in the glass be represented by

$$\epsilon^{\iota\kappa_1(-x\cos i+y\sin i-V_1t)},$$

where $\kappa_1 = 2\pi/\lambda_1$, λ_1 being the wave-length in glass. Then the reflected vibration will be represented by

$$b\epsilon^{\iota\kappa_1(x\cos i+y\sin i-V_1t)}.$$

[1] Report on Physical Optics, *Brit. Assoc. Rep.*, vol. III. p. 310.

[2] *Trans. Camb. Phil. Soc.* vol. VIII. p. 642; *Math. and Phy. Papers*, vol. II. p. 56.

Since the refracted wave is a superficial wave, it may be represented by

$$c\epsilon^{a_1 x + \iota\kappa(y \sin i - Vt)},$$

where $\kappa = 2\pi/\lambda$, λ being the wave-length in air.

The quantities b and c are complex quantities, and their values as well as the value of a_1, will depend partly upon the particular dynamical theory which we adopt, and partly also upon whether the light is polarized in or perpendicularly to the plane of incidence.

When the superficial wave is reflected at the second surface, the amplitude of the reflected wave will be ceq, where $q = \epsilon^{-a_1 D}$. This wave will be again reflected and refracted at the first surface, and the amplitudes of the two vibrations will be ce^2q^2 and $cefq^2$ respectively; whence taking into account the infinite series of reflections, we obtain for the amplitude of the vibration which is finally refracted into the glass, the expression

$$b + cefq^2(1 + e^2q^2 + e^4q^4 + \ldots\ldots) = b + \frac{cefq^2}{1 - e^2q^2}.$$

Since by (1) of § 26,

$$b = -e, \quad cf = 1 - e^2,$$

the reflected wave becomes

$$\frac{(1 - q^2)\,b}{1 - q^2b^2}\,\epsilon^{\iota\kappa_1(x \cos i + y \sin i - V_1 t)} \quad \ldots\ldots\ldots\ldots\ldots(48).$$

201. It will now be necessary to distinguish between light polarized in, and light polarized perpendicularly to the plane of incidence. Taking the first, it follows from Fresnel's theory that

$$a_1 = (2\pi/\lambda)\,(\mu^2 \sin^2 i - 1)^{\frac{1}{2}},$$

where λ is the wave-length in *air*, and μ is the index of refraction from air to glass. Also from the same theory, see § 173,

$$b = \cos 2\pi e/\lambda_1 - \iota \sin 2\pi e/\lambda_1 = \epsilon^{-2\iota\pi e/\lambda_1},$$

where

$$\tan\frac{\pi e}{\lambda_1} = \frac{(\mu^2 \sin^2 i - 1)^{\frac{1}{2}}}{\mu \cos i} \quad \ldots\ldots\ldots\ldots\ldots\ldots(49).$$

Writing $\theta = \pi e/\lambda_1$, the coefficient in (48) may be written

$$\frac{1 - q^2}{\epsilon^{2\iota\theta} - q^2\epsilon^{-2\iota\theta}} = \frac{1 - q^2}{(1 - q^2)\cos 2\theta + \iota(1 + q^2)\sin 2\theta}$$

$$= \frac{(1 - q^2)\{(1 - q^2)\cos 2\theta - \iota(1 + q^2)\sin 2\theta\}}{(1 - q^2)^2 + 4q^2 \sin^2 2\theta},$$

and therefore the intensity is proportional to

$$\rho^2 = \frac{(1-q^2)^2}{(1-q^2)^2 + 4q^2 \sin^2 2\theta} \quad \text{..............(50)},$$

where $$q = exp\,\{-(2\pi D/\lambda)(\mu^2 \sin^2 i - 1)^{\frac{1}{2}}\} \quad \text{...........(51)}.$$

202. When the light is polarized perpendicularly to the plane of incidence, we must write θ' for θ, where

$$\tan \theta' = \mu (\mu^2 \tan^2 i - \sec^2 i)^{\frac{1}{2}} \quad \text{..............(52)}.$$

In either case the reflected vibration is accordingly equal to

$$\rho \cos \frac{2\pi}{\lambda_1}(x \cos i + y \sin i - V_1 t - \psi),$$

where $$\tan \frac{2\pi\psi}{\lambda_1} = \frac{1+q^2}{1-q^2} \tan 2\phi \quad \text{..............(53)},$$

and ϕ is equal to θ or θ', according as the light is polarized in or perpendicularly to the plane of incidence.

203. The corresponding formulæ for the transmitted light can be obtained in a similar manner. When the superficial wave arrives at the second surface, its amplitude is equal to cq; the amplitude of the refracted portion is cqf, and that of the reflected portion is cqe; whence after one reflection at the first plate, and one refraction at the second plate, the portion refracted at the latter becomes cq^3e^2f. Hence the amplitude of the refracted portion is

$$cqf(1 + e^2q^2 + e^4q^4 + \ldots) = \frac{cqf}{1 - e^2q^2} = \frac{q(b^{-1} - b)}{b^{-1} - q^2 b}.$$

Substituting the value of b, this becomes

$$\frac{2\iota q \sin 2\theta}{(1-q^2)\cos 2\theta + \iota(1+q^2)\sin 2\theta}$$
$$= \frac{2q \sin 2\theta \{(1+q^2)\sin 2\theta + \iota(1-q^2)\cos 2\theta\}}{(1-q^2)^2 + 4q^2 \sin^2 2\theta}.$$

Whence the intensity of the transmitted light is

$$\rho_1^2 = \frac{4q^2 \sin^2 2\theta}{(1-q^2)^2 + 4q^2 \sin^2 2\theta} \quad \text{..............(54)},$$

and the transmitted wave is

$$\rho_1 \cos \frac{2\pi}{\lambda_1}(x \cos i - y \sin i + V_1 t - \psi_1),$$

where $$\tan \frac{2\pi\psi_1}{\lambda_1} = \frac{1-q^2}{1+q^2} \cot 2\theta \quad \text{................ (55)}.$$

In equations (54) and (55), $\tan\theta$ is determined by the right-hand sides of (49) or (52), according as the light is polarized in or perpendicularly to the plane of incidence.

204. In examining these expressions, we shall suppose that the rings are formed by a lens in contact with a prism or a plate of glass. Hence at the point of contact, $D=0$, and therefore by (51) $q=1$; accordingly $\rho^2=0$, or there is absolute darkness. On receding from the point of contact, q decreases, but slowly at first, inasmuch as $D \propto r^2$, where r is the distance from the point of contact; whence the intensity ultimately varies as r^4, and therefore increases with extreme slowness. There is consequently a black spot at the centre. Further on q decreases rapidly, and soon becomes insensible; hence as we proceed from the black spot, the intensity at first increases rapidly, and then slowly again as it approaches its limiting value unity, to which it soon becomes sensibly equal.

205. We shall now consider how the intensity varies with the colour. Since μ is a function of λ, it follows that θ and θ' depend upon the colour, but the variations in passing from one colour to another are so small, that they may be left out of account. From (51) we see, that as the distance from the point of contact increases, q decreases more rapidly when λ is small; accordingly the spot must be smaller for blue light than for red light, and therefore the black spot must be bordered by a ring of blue light. On the other hand towards the edge of the bright spot, which is seen by transmission, the colours at the red end predominate.

206. We shall next consider, how the spot depends upon the nature of the polarization. Let s be the ratio of the transmitted light to the reflected; s_1, s_2 the particular values of s, belonging to light polarized in and perpendicularly to the plane of incidence; then

$$s_1 = \frac{4q^2 \sin^2 2\theta}{(1-q^2)^2}, \quad s_2 = \frac{4q^2 \sin^2 2\theta'}{(1-q^2)^2},$$

whence

$$\frac{s_1}{s_2} = \frac{\sin^2 2\theta}{\sin^2 2\theta'} = \{(\mu^2+1)\sin^2 i - 1\}^2 \ldots\ldots\ldots\ldots (56).$$

Now the distinctness of the black spot, which is produced by reflection, depends upon s being large; and since in the neighbourhood of the critical angle, we have from (56), $s_2 = s_1\mu^4$, it follows that the spot is much more conspicuous for light polarized

perpendicularly to the plane of incidence, than for light polarized in that plane. As i increases, the spots seen in the two cases become more and more nearly equal, and become exactly alike when $i = i'$, where $\operatorname{cosec}^2 i' = \frac{1}{2}(1 + \mu^2)$. When i becomes greater than i', the order of magnitude is reversed, and when $i = \frac{1}{2}\pi$, $s_1 = s_2\mu^4$, so that the inequality becomes large. It must however be recollected, that this statement refers to the relative magnitudes of the spots, for when the angle of incidence is nearly equal to $\frac{1}{2}\pi$, the absolute magnitudes of the spots become very small.

All the conclusions deduced by the above theory have been verified experimentally by Stokes, and he has also discussed the case in which the incident light is polarized in a plane, making a given angle with the plane of incidence.

The Intensity of Light reflected from a Pile of Plates.

207. A common method of producing polarized light is by means of a pile of plates, and we shall now give an investigation, due to Stokes, of the intensity of the light reflected from, or transmitted through a pile of plates[1].

The plates will be supposed to be formed of the same material, to be of equal thickness, and to be placed parallel to one another, and the plates and the intermediate layers of air, to be sufficiently thick to prevent the occurrence of the colours of thin plates.

In consequence of absorption, the intensity of light traversing an elementary distance dx of a plate, will be reduced in the proportion of 1 to $1 - q\,dx$, where q is a constant. Hence if I, $I + dI$ be the intensities of the light on entering and emerging from a layer, whose thickness is dx,

$$dI = -Iq\,dx \quad\quad (57),$$

whence if τ is the thickness of the plate, r the angle which the direction of the light makes with the normal to the plate, and I_0, I_1 the intensities on entrance and emergence, we obtain from (57)

$$I_1 = I_0\,\epsilon^{-\tau q \sec r} = I_0 g \quad \text{(say)}.$$

The constant q, which may be called the coefficient of absorption, depends upon the material of which the plate is made; also since the amount of light absorbed varies with its refrangibility, q also depends upon the colour.

[1] Stokes, *Proc. Roy. Soc.* vol. XI. p. 545.

208. Let ρ be the fraction of the light, which is reflected at the first surface of a plate; then $1-\rho$ is the fraction of the light which is transmitted.

Since the light reflected by a plate is made up of that which is reflected at the first surface, and that which has suffered an odd number of internal reflections, it follows that if the intensity of the incident light be taken as unity, the intensity of these various portions will be

$$\rho,\ (1-\rho)^2\rho g^2,\ (1-\rho)^2\rho^3 g^4, \text{ etc.}$$

Hence if R be the intensity of the reflected light,

$$\begin{aligned} R &= \rho + (1-\rho)^2\rho g^2 (1 + \rho^2 g^2 + \rho^4 g^4 + \ldots) \\ &= \rho + \frac{(1-\rho)^2 \rho g^2}{1-\rho^2 g^2} \quad\ldots\ldots\ldots\ldots(58). \end{aligned}$$

Similarly if T be the intensity of the transmitted light,

$$T = \frac{(1-\rho)^2 g}{1-\rho^2 g^2} \quad\ldots\ldots\ldots\ldots(59).$$

209. The value of ρ depends upon the particular theory of light which we adopt, but in any case it may be supposed to be a known function of i the angle of incidence and μ the index of refraction; the value of g depends upon i and μ, and also upon q, which may be supposed to have been determined by experiment. To complete the solution, we have therefore to solve the following problem:—*There are* m *parallel plates, each of which reflects and transmits given fractions* R *and* T *of the light incident upon it; light of intensity unity being incident upon the system, it is required to find the intensities of the reflected and refracted light.*

Let these be denoted by $\phi(m)$, $\psi(m)$; and consider a system of $m+n$ plates, and imagine them grouped into two systems of m and n plates respectively. Since the incident light is represented by unity, $\phi(m)$ will be the intensity of the light reflected from the first group, whilst $\psi(m)$ will be transmitted. A fraction $\phi(n)$ of the latter will be reflected by the second group, whilst a portion $\psi(n)$ will be transmitted; and the fraction $\phi(m)$ of the light reflected by the second group will be reflected by the first group, whilst the fraction $\psi(m)$ will be transmitted, and so on. It therefore follows, that the intensity of the light reflected by the whole system will be

$$\phi(m) + (\psi m)^2 \phi(n) + (\psi m)^2 \phi(m)\, \phi(n)^2 + \ldots,$$

and the intensity of the light transmitted will be

$$\psi(m)\psi(n)+\psi(m)\phi(n)\phi(m)\psi(n)+\psi(m)(\phi n)^2(\phi m)^2\psi(n)+\dots$$

The first of these expressions is equal to $\phi(m+n)$, whilst the second is equal to $\psi(m+n)$; whence summing the two geometrical series, we obtain

$$\phi(m+n)=\phi(m)+\frac{(\psi m)^2\phi(n)}{1-\phi(m)\phi(n)}\quad\dots\dots\dots(60),$$

$$\psi(m+n)=\frac{\psi(m)\psi(n)}{1-\phi(m)\phi(n)}\quad\dots\dots\dots\dots\dots\dots(61).$$

In the special problem under consideration, m and n are positive integers; but we shall now show how to obtain the solution of these two functional equations, when m and n have any values whatever. From (60) we obtain

$$\phi(m+n)\{1-\phi(m)\phi(n)\}=\phi(m)+\phi(n)\{(\psi m)^2-(\phi m)^2\}.$$

Since the left-hand side of this equation is symmetrical with respect to m and n, we obtain by interchanging these letters, and equating the results

$$\frac{1+(\phi m)^2-(\psi m)^2}{\phi(m)}=\frac{1+(\phi n)^2-(\psi n)^2}{\phi(n)}.$$

Since this equation is true for all values of m and n, each side must be equal to a constant; whence denoting the constant by $2\cos\alpha$, we obtain

$$(\psi m)^2=1-2\phi(m)\cos\alpha+(\phi m)^2\quad\dots\dots\dots\dots(62).$$

Squaring (61), and eliminating the function ψ by means of (62), we obtain

$$\{1-\phi(m)\phi(n)\}^2[1-2\phi(m+n)\cos\alpha+\{\phi(m+n)\}^2]$$
$$=\{1-2\phi(m)\cos\alpha+(\phi m)^2\}\{1-2\phi(n)\cos\alpha+(\phi n)^2\}\dots(63).$$

In order that (60) and (61) may hold good for a zero value of one of the variables, say n, we must have $\phi(0)=0$, $\psi(0)=1$. If however we put $n=0$ in (63), the equation reduces to an identity; we must therefore differentiate (63) with respect to n, and then put $n=0$. Accordingly we find

$$\phi'(0)\phi(m)\{1-2\phi(m)\cos\alpha+(\phi m)^2\}+\phi'(m)\cos\alpha-\phi(m)\phi'(m)$$
$$=\{1-2\phi(m)\cos\alpha+(\phi m)^2\}\phi'(0)\cos\alpha.$$

Dividing out by $\phi(m)-\cos\alpha$, since the solution $\phi(m)=\cos\alpha$ would lead to $\psi(m)=C$, we obtain

$$\phi'(m)=\phi'(0)\{1-2\phi(m)\cos\alpha+(\phi m)^2\}\quad\dots\dots\dots(64).$$

Integrating this equation, and determining the arbitrary constant from the condition that $\phi(0)=0$, and writing β for $\phi'(0)\sin\alpha$, we obtain

$$\phi(m)=\frac{\sin m\beta}{\sin(\alpha+m\beta)}\dots\dots\dots\dots\dots\dots(65).$$

Substituting in (61) and reducing, we find

$$\psi(m)=\frac{\sin\alpha}{\sin(\alpha+m\beta)}\dots\dots\dots\dots\dots\dots(66).$$

Equations (65) and (66) may be written in the form

$$\frac{\phi(m)}{\sin m\beta}=\frac{\psi(m)}{\sin\alpha}=\frac{1}{\sin(\alpha+m\beta)}\dots\dots\dots\dots(67).$$

When $m=1$, $\phi(m)=R$, $\psi(m)=T$, where the values of R and T are given by (58) and (59); and therefore to determine the arbitrary constants, we have the equations

$$\frac{R}{\sin\beta}=\frac{T}{\sin\alpha}=\frac{1}{\sin(\alpha+\beta)}\dots\dots\dots\dots\dots\dots(68).$$

210. Equations (67) and (68) give the following quasi-geometrical construction for solving the problem:—*Construct a triangle, in which the sides represent in magnitude the intensities of the incident, reflected and transmitted light in the case of a single plate; then leaving the first side and the angle opposite to the third unchanged, multiply the angle opposite the second, by the number of plates; then the sides of the new triangle will represent the corresponding intensities in the case of a system of plates.* This construction cannot however be actually effected, inasmuch as the first side of the triangle is greater than the sum of the two others, and the angles are therefore imaginary.

To adapt the formulæ to numerical calculation, it will be convenient to get rid of the imaginary quantities. Putting

$$\{(1+R+T)(1+R-T)(1+T-R)(1-R-T)\}^{\frac{1}{2}}=\Delta\dots(69),$$

we have by ordinary Trigonometry

$$\cos\alpha=\frac{1+R^2-T^2}{2R},\quad \sin\alpha=\pm\frac{\iota\Delta}{2R};$$

whence putting $\quad (1+R^2-T^2+\Delta)/2R=a\dots\dots\dots\dots\dots\dots(70),$

we have $\quad \epsilon^{\iota\alpha}=\cos\alpha+\iota\sin\alpha=a^{\mp1}.$

Choosing the lower signs, we have

$$2R\sin\alpha = -\iota\Delta, \quad \epsilon^{\iota\alpha} = a;$$

also $$\cos\beta = \frac{1+T^2-R^2}{2T}, \quad \sin\beta = \frac{R\sin\alpha}{T} = -\frac{\iota\Delta}{2T}.$$

Whence if $$(1+T^2-R^2+\Delta)/2T = b \quad\text{..............(71)},$$

we shall have $$\epsilon^{\iota\beta} = b,$$

and (67) becomes

$$\frac{\phi(m)}{b^m - b^{-m}} = \frac{\psi(m)}{a - a^{-1}} = \frac{1}{ab^m - a^{-1}b^{-m}} \quad\text{............(72)}.$$

211. From this equation we see, that the intensity of the light reflected from an infinite number of plates is a^{-1}; and since a is changed into a^{-1}, by changing the sign of α or Δ, we have

$$a^{-1} = (1+R^2+T^2-\Delta)/2R \quad\text{..............(73)},$$

which is equal to unity in the case of perfect transparency. Accordingly substances, such as snow and colourless compounds thrown down as chemical precipitates, which are finely divided so as to present numerous reflecting surfaces, and which are transparent in mass, are brilliantly white by reflected light.

212. The following tables, taken from Stokes' paper, give the intensity of the light reflected from, or transmitted through, a pile of m plates for the values 1, 2, 4 and ∞ of m for three degrees of transparency, and for certain selected angles of incidence. The refractive index is taken to be equal to 1·52; $\delta = 1 - \epsilon^{-qT}$ is the loss by absorption in a single transit through a plate at perpendicular incidence, so that $\delta = 0$ corresponds to perfect transparency; also the value of ρ is supposed to be calculated from Fresnel's formulæ, so that

$$\rho = \frac{\sin^2(i-r)}{\sin^2(i+r)} \text{ or } \frac{\tan^2(i-r)}{\tan^2(i+r)} \quad\text{..............(74)},$$

according as the light is polarized in or perpendicularly to the plane of incidence. The angle ϖ is the polarizing angle $\tan^{-1}\mu$; ϕ and ψ denote the intensities of the reflected and transmitted light, the intensity of the incident light being taken as 1000. For oblique incidences, it is necessary to distinguish between

light polarized in and perpendicularly to the plane of incidence, and the suffixes 1 and 2 refer to these two kinds respectively.

I.

	$\delta=0$								
	i					$i=\varpi+2^{\circ}$			
m	ϕ		ϕ_1	ψ_1		ψ_1	ϕ_2		
1	82	918	271	729	300	700	1	999	·701
2	151	849	426	574	459	541	2	998	·542
4	262	738	598	402	628	372	4	996	·373
8	416	584	749	251	771	229	8	992	·231
16	587	413	856	144	870	130	16	984	·132
32	740	260	922	78	931	69	32	968	·071
∞	1000	0	1000	0		0	1000	0	·000

II.

	$\delta=\cdot02$						$\delta=\cdot1$					
	$i=0$						$i=0$					
			ϕ_1		ψ_2	ψ_1/ψ_2	ϕ		ϕ_1	ψ_1	ψ_2	ψ_1/ψ_2
1	80	900	265	711	976	·728	74	826	245	639	881	·725
2	145	815	410	544	953	·571	125	686	351	435	777	·559
4	244	679	555	355	908	·391	185	479	427	215	604	·357
8	364	490	656	182	824	·221	229	237	451	57	365	·156
16	464	276	695	58	679	·086	243	59	453	4	133	·030
32	509	97	699	7	461	·014	244	4	453	0	18	·001
∞	1000	0	699	0	0	·000	244	0	453		0	·000

213. In discussing these tables Sir G. Stokes says :—"The intensity of the light reflected from a pile consisting of an infinite number of similar plates, falls off rapidly with the transparency of the material of which the plates are composed, especially at small incidence. Thus at a perpendicular incidence, we see from the above table that the reflected light is reduced to little more than one half, when 2 per cent. is absorbed in a single transit; and to less than a quarter, when 10 per cent. is absorbed.

"With imperfectly transparent plates, little is gained by multiplying the plates beyond a very limited number, if the object be to obtain light, as bright as may be, polarized by reflection. Thus the table shows, that 4 plates of the less

defective kind (for which $\delta = \cdot 02$) reflect 79 per cent.; and 4 plates of the more defective kind (for which $\delta = \cdot 1$) reflect as much as 94 per cent. of the light, that could be reflected by a greater number; whereas 4 plates of the perfectly transparent kind reflect only 60 per cent.

"The table also shows, that while the amount of light transmitted at the polarizing angle by a pile of a considerable number of plates is materially reduced by a defect of transparency, its state of polarization is somewhat improved. This result might be seen without calculation. For while no part of the transmitted light which is polarized perpendicularly to the plane of incidence underwent reflection, a large part of the transmitted light polarized the other way was reflected an even number of times; and since the length of path of the light within the absorbing medium is necessarily increased by reflection, it follows that a defect of transparency must operate more powerfully in reducing the intensity of light polarized in, than of light polarized perpendicularly to the plane of polarization. But the table also shows, that a far better result can be obtained, as to the perfection of the polarization of the transmitted light, without any greater loss of illumination, by employing a larger number of plates of the more transparent kind."

214. We shall now confine our attention to perfectly transparent plates, and consider the manner in which the degree of polarization of the *transmitted* light varies with the angle of incidence.

The degree of polarization is expressed by the ratio ψ_1/ψ_2, which we shall denote by χ. When $\chi = 1$, there is no polarization; and when $\chi = 0$, the polarization is perfect in a plane perpendicular to the plane of incidence. Now when $g = 1$, it follows from (58) and (59) that

$$R = \frac{2\rho}{1+\rho}, \qquad T = \frac{1-\rho}{1+\rho},$$

whence $R + T = 1$; accordingly from (70) and (71), a and b are each equal to unity, and (72) becomes indeterminate. Now when a and b are nearly equal to unity, α and β become indefinitely small, whence (67) becomes

$$\frac{\phi(m)}{m\beta} = \frac{\psi(m)}{\alpha} = \frac{1}{\alpha + m\beta}.$$

Also from (68), $\beta = R\alpha/T$, whence

$$\frac{\phi(m)}{2m\rho} = \frac{\psi(m)}{1-\rho} = \frac{1}{1+(2m-1)\rho} \quad\text{..............(75).}$$

Let $\qquad i - r = \theta, \quad i + r = \sigma, \quad \mu \sin r = \sin i,$

then

$$\frac{di}{\tan i} = \frac{dr}{\tan r} = \frac{d\theta}{\tan i - \tan r} = \frac{d\sigma}{\tan i + \tan r} = \cos i \cos r d\omega \text{ (say)};$$

whence $\qquad d\theta = \sin\theta d\omega, \quad d\sigma = \sin\sigma d\omega$ ....(76),

and we see that i and ω increase together, from $i=0$ to $i=\frac{1}{2}\pi$. We also obtain from (74)

$$\rho_1 = \frac{\sin^2\theta}{\sin^2\sigma},$$

$$d\rho_1 = 2\sin\theta \operatorname{cosec}^3\sigma(\sin\sigma\cos\theta d\theta - \sin\theta\cos\sigma d\sigma)$$
$$= 2\sin^2\theta \operatorname{cosec}^2\sigma(\cos\theta - \cos\sigma)\,d\omega.$$

Also $\qquad \rho_2 = \dfrac{\tan^2\theta}{\tan^2\sigma},$

$$d\rho_2 = 2\tan\theta\cot^3\sigma(\tan\sigma\sec^2\theta d\theta - \tan\theta\sec^2\sigma d\sigma)$$
$$= \frac{2\sin^2\theta\cos\sigma}{\cos^3\theta\sin^2\sigma}(\cos\sigma - \cos\theta)\,d\omega$$
$$= -\frac{\cos\sigma}{\cos^3\theta}d\rho_1.$$

Now $\cos\theta - \cos\sigma$ or $2\sin i \sin r$ is positive; and $\cos\sigma$ is positive from $i=0$ to $i=\varpi$, and negative from $i=\varpi$ to $i=\frac{1}{2}\pi$. But (76) shows that ψ decreases as ρ increases. From $i=0$ to $i=\varpi$, ρ_1 increases and ρ_2 decreases, and therefore on both accounts χ decreases. When $i=\varpi$, $d\rho_1/di$ is still positive, and therefore $d\psi_1/di$ is negative; but the maximum value of ψ_2 is 1, so that on passing through the polarizing angle, χ still decreases, or the polarization improves. When the plates are very numerous, $\psi_2 = 1$ at the polarizing angle, and on both sides of it decreases rapidly; whereas ψ_1, which is always small, suffers no particular change about the polarizing angle. Hence in this case, χ must be a minimum a little beyond the polarizing angle. In order to find the angle of incidence, which makes χ a minimum in the case of an arbitrary number of plates, we have from (74) and (75)

$$\chi = \frac{(\sin^2\sigma - \sin^2\theta)\{\sin^2\sigma\cos^2\theta + (2m-1)\sin^2\theta\cos^2\sigma\}}{\{\sin^2\sigma + (2m-1)\sin^2\theta\}(\sin^2\sigma\cos^2\theta - \sin^2\theta\cos^2\sigma)}$$
$$= 1 - \frac{2m}{\operatorname{cosec}^2\theta + (2m-1)\operatorname{cosec}^2\sigma} \quad\text{......................(77).}$$

Hence χ is a minimum along with the denominator. Differentiating and taking account of (76), we obtain the following equation to determine the angle of maximum polarization, viz.

$$\cos\theta\sin^2\sigma + (2m-1)\cos\sigma\sin^2\theta = 0 \qquad\qquad (78).$$

For any assumed value of i from ϖ to $\frac{1}{2}\pi$, this equation gives the value of m, that is the number of plates of which the pile must be composed, in order that the assumed incidence may be that of maximum polarization of the transmitted light. The equation may be put into the form

$$2m-1 = -\frac{\tan\sigma\sin\sigma}{\tan\theta\sin\theta} = \frac{1}{(\rho_1\rho_2)^{\frac{1}{2}}}.$$

Now we have seen that both ρ_1 and ρ_2 continually increase, and therefore m continually decreases from $i=\varpi$ to $i=\frac{1}{2}\pi$. At the first of these limits $\rho_2=0$, and therefore $m=\infty$; at the second, $\rho_1=\rho_2=1$, and therefore $m=1$. Hence with a single plate, the polarization of the transmitted light continually improves up to a grazing incidence; but with a pile of plates, the polarization attains a maximum at an angle of incidence, which approaches indefinitely to the polarizing angle, as the number of plates is indefinitely increased.

Eliminating m from (77) and (78), we find

$$\chi = -\cos\theta\cos\sigma \qquad\qquad (79),$$

which determines for any pile χ_1, the defect of maximum polarization of the transmitted light, in terms of the angle of incidence, for which the polarization is a maximum. We have from (79), (76) and (78)

$$d\chi_1 = (\sin^2\theta\cos\sigma + \sin^2\sigma\cos\theta)\,d\omega$$
$$= -2(m-1)\cos\sigma\sin^2\theta\,d\omega;$$

and since $\cos\sigma$ is negative, when the angle of incidence exceeds the polarizing angle, it follows that χ_1 decreases as ω (and therefore i) decreases, or as m increases. For $m=1$, $i=\frac{1}{2}\pi$, and $\chi_1=\mu^{-2}$; for $m=\infty$, $\cos\sigma=0$, and have $\chi_1=0$; or the maximum polarization tends to become perfect, as the number of plates is indefinitely increased.

CHAPTER XIII.

DYNAMICAL THEORY OF DIFFRACTION.

215. In the present Chapter, we shall apply Green's theory to the problem of diffraction, and we shall commence by investigating the propagation of an arbitrary disturbance.

If we put $a^2 = (k + \frac{4}{3}n)/\rho, \quad b^2 = n/\rho$;

so that a and b are the velocities of propagation of the dilatational and distortional waves, the equations of motion (12) of § 187, may be written

$$\left.\begin{aligned}\frac{d^2u}{dt^2} &= (a^2 - b^2)\frac{d\delta}{dx} + b^2\nabla^2 u\\ \frac{d^2v}{dt^2} &= (a^2 - b^2)\frac{d\delta}{dy} + b^2\nabla^2 v\\ \frac{d^2w}{dt^2} &= (a^2 - b^2)\frac{d\delta}{dz} + b^2\nabla^2 w\end{aligned}\right\} \text{.(1).}$$

Since the dilatational and distortional waves are propagated independently, we may divide u, v, w into two parts, viz. u_1, v_1, w_1 and u_2, v_2, w_2, of which the first depends upon the dilatation, and the second upon the distortion. Since the rotations ξ, η, ζ are zero when there is no distortion, it follows from (7) of § 187, that $dw_1/dy - dv_1/dz$ &c. &c. are zero, and therefore $u_1dx + v_1dy + w_1dz$ is a perfect differential $d\phi$; we may therefore put

$$u_1 = \frac{d\phi}{dx}, \quad v_1 = \frac{d\phi}{dy}, \quad w_1 = \frac{d\phi}{dz} \text{..............(2)},$$

where ϕ is some function of x, y, z and t.

Since the displacements u_2, v_2, w_2 do not involve dilatation, it follows that

$$\frac{du_2}{dx} + \frac{dv_2}{dy} + \frac{dw_2}{dz} = 0 \text{....................(3)}.$$

From (1) we see that

$$\frac{d^2\phi}{dt^2} = a^2\nabla^2\phi \quad\text{.........................(4),}$$

also putting $u = d\phi/dx + u_2$ in (1), and taking account of (4), we obtain

$$\frac{d^2u_2}{dt^2} = b^2\nabla^2u_2 \quad\text{....................(5),}$$

with two similar equations for v_2, w_2. It therefore follows that the most general solution of (1) is

$$u = \frac{d\phi}{dx} + u_2, \quad v = \frac{d\phi}{dy} + v_2, \quad w = \frac{d\phi}{dz} + w_2 \quad\text{...........(6),}$$

where ϕ is any function of x, y, z and t which satisfies (4), and u_2, v_2, w_2 are similar functions, which satisfy equations of the form (5), subject to equation (3).

Propagation of an Arbitrary Disturbance.

216. We shall now apply equations (1) to obtain the solution of a problem which was first investigated by Sir G. Stokes, viz. the propagation of an arbitrary disturbance in an elastic medium[1].

Let us suppose that the medium is initially at rest, and that a disturbance is excited throughout a certain volume T of the medium. The subsequent character of the disturbance is completely determined, when the initial displacement and the initial velocity of every element within T is known. Let P be any point within T, O the point at which the disturbance is sought, and let us first consider that portion of the disturbance which depends upon the dilatation δ.

By equations (3) and (6), it follows that $\delta = \nabla^2\phi$, where ϕ satisfies (4); and it will be more convenient to consider the function ϕ than δ, for when the former function is known, the portions of the displacements which depend upon the dilatation can be immediately obtained by differentiation.

If x, y, z be the coordinates of O, the initial values of ϕ and $\dot{\phi}$, will depend entirely upon the position of O, and we must therefore have

$$\phi_0 = f(x, y, z), \quad \dot{\phi}_0 = F(x, y, z) \quad\text{..............(7),}$$

[1] *Trans. Camb. Phil. Soc.* Vol. IX. p. 1, and *Math. and Phys. Papers*, Vol. II. p. 243.

where f and F are given functions. We therefore require the solution of (4) subject to (7).

217. The solution of (4), which was first obtained by Poisson, may be effected as follows. The symbolic solution is

$$\phi = \cosh(at\nabla)\chi + \sinh(at\nabla)\psi,$$

where χ and ψ are functions of x, y, z; we therefore obtain from (7)

$$f = \phi_0 = \chi,$$

and $$F = \dot{\phi}_0 = a\nabla\psi.$$

Accordingly the solution becomes

$$\phi = \cosh(at\nabla)f + \frac{\sinh(at\nabla)}{a\nabla}F \quad\text{..............(8).}$$

We must now show how the operations denoted by the symbolic operators may be performed.

With the point O as a centre, describe a sphere of radius r, and let α, β, γ be the coordinates of any point P on this sphere relatively to O; also let us temporarily denote d/dx, d/dy, d/dz by λ, μ, ν.

Consider the integral

$$\iint \epsilon^{\lambda\alpha+\mu\beta+\nu\gamma}\, dS,$$

where the integration extends over the surface of the sphere. If through the point α, β, γ, a plane be drawn, whose direction cosines are proportional to λ, μ, ν, and if p be the perpendicular from O on to this plane, it is known that

$$\lambda\alpha + \mu\beta + \nu\gamma = (\lambda^2 + \mu^2 + \nu^2)^{\frac{1}{2}} p = \nabla p.$$

Also if θ be the angle, which the radius drawn from O to the point P makes with p, then $p = r\cos\theta$, $dS = 2\pi r^2 \sin\theta d\theta$; whence the integral

$$= 2\pi r^2 \int_0^\pi \epsilon^{\nabla r\cos\theta} \sin\theta d\theta$$

$$= 4\pi r^2 \frac{\sinh(r\nabla)}{r\nabla}.$$

But if $d\Omega$ be the elementary solid angle subtended by dS at O, $dS = r^2 d\Omega$; whence putting $r = at$, and restoring the values of λ, μ, ν, we obtain

$$\frac{\sinh(at\nabla)}{a\nabla} = \frac{t}{4\pi}\iint \epsilon^{\alpha\frac{d}{dx}+\beta\frac{d}{dy}+\gamma\frac{d}{dz}}\, d\Omega.$$

Now the operation denoted by the exponential factor on the right-hand side of the last equation, can be performed by means of the symbolic form of Taylor's theorem; we thus obtain

$$\frac{\sinh(at\nabla)}{a\nabla}F(x, y, z) = \frac{t}{4\pi}\iint \epsilon^{\alpha\frac{d}{dx}+\beta\frac{d}{dy}+\gamma\frac{d}{dz}} F(x, y, z)\, d\Omega$$

$$= \frac{t}{4\pi}\iint F(x+\alpha, y+\beta, z+\gamma)\, d\Omega.$$

If l, m, n be the direction cosines of OP, we shall have $\alpha = lat$, $\beta = mat$, $\gamma = nat$; and therefore the portion of ϕ which depends upon the initial velocities is

$$\frac{t}{4\pi}\iint F(x + lat, y + mat, z + nat)\, d\Omega.$$

From the form of (8) it is at once seen, that the portion of ϕ depending on the initial displacements may be obtained by changing F into f, and differentiating with respect to t; we thus obtain

$$\phi = \frac{t}{4\pi}\iint F(x + lat, y + mat, z + nat)\, d\Omega,$$

$$+\frac{1}{4\pi}\frac{d}{dt}t\iint f(x + lat, y + mat, z + nat)\, d\Omega \quad\ldots\ldots\ldots(9).$$

This equation determines the value of ϕ at time t, at any point O of the medium whose coordinates are x, y, z, in terms of the initial values of ϕ and $\dot{\phi}$. The portions of the displacements which depend upon the dilatation are obtained by differentiating (9) with respect to x, y, z.

218. If the initial disturbance is confined to a portion T of the medium, the double integrals in (9) will vanish, unless the sphere whose centre is O and whose radius is at cuts a portion of the space T. Hence if O be outside T, and if r_1, r_2 be respectively the least and greatest values of the radius vector of any element of that space, there will be no dilatation at O until $at = r_1$. The dilatation at O will then commence, and will last during an interval $(r_2 - r_1)/a$, and will then cease for ever.

219. If f_1, f_2, f_3 denote the initial values of u_2, v_2, w_2, which are the portions of the displacements which depend upon the distortion, and if F_1, F_2, F_3 denote the initial values of $\dot{u}_2$, $\dot{v}_2$, $\dot{w}_2$, then since u_2, v_2, w_2 each satisfy equations of the same form as (4) with b written for a, it follows that the values of these quantities

at time t are determined by equations of the same form as (9). It must also be recollected that f_1, f_2, f_3 and also F_1, F_2, F_3 satisfy (3).

If therefore we write for brevity

$$F(at) \text{ for } F(x + lat,\ y + mat,\ z + nat),$$

the complete value of u will be

$$u = \frac{t}{4\pi}\iint \frac{d}{dx} F(at)\, d\Omega + \frac{1}{4\pi}\frac{d}{dt} t \iint \frac{d}{dx} f(at)\, d\Omega,$$

$$+ \frac{t}{4\pi}\iint F_1(bt)\, d\Omega + \frac{1}{4\pi}\frac{d}{dt} t \iint f_1(bt)\, d\Omega \quad\text{.........(10)},$$

with similar expressions for v and w.

220. The initial velocities are determined by the equations

$$\dot{u}_0 = \frac{dF}{dx} + F_1,$$

$$\dot{v}_0 = \frac{dF}{dy} + F_2,$$

$$\dot{w} = \frac{dF}{dz} + F_3,$$

where F_1, F_2, F_3 satisfy (3); and since our object is to find the values of u, v, w at any subsequent time in terms of the values of the initial displacements and velocities, we must proceed to eliminate the F's and f's from (10). It will however be sufficient to perform this operation for those parts of u, v, w which depend upon the initial velocities, for when this is done, the portions depending upon the initial displacements can be obtained by differentiating with respect to t and changing $\dot{u}_0$, $\dot{v}_0$, $\dot{w}_0$ into u_0, v_0, w_0.

221. Let α, β, γ denote the coordinates of any point P relatively to O; let $OP = r$, and let l, m, n be the direction cosines of OP; then at points on the surface of the sphere $r = at$, we have $\alpha = lat$, &c.; also if

$$\chi = \frac{d}{dx} F(at) \quad\text{.........................(11)},$$

the first term of (10) becomes

$$\frac{t}{4\pi}\iint \chi\, d\Omega = \frac{t}{4\pi}\iint \frac{\chi}{r^2}\, dS \quad\text{..................(12)}.$$

By Green's Theorem,

$$\iint \phi \frac{d\psi}{d\nu}\, dS \pm \iiint \phi \nabla^2 \psi\, d\alpha d\beta d\gamma = \iint \psi \frac{d\phi}{d\nu}\, dS \pm \iiint \psi \nabla^2 \phi\, d\alpha d\beta d\gamma$$

where $\nabla^2 = d^2/d\alpha^2 + d^2/d\beta^2 + d^2/d\gamma^2$, $d\nu$ is an element of the normal to the sphere $r = at$, and the upper or lower sign is to be taken, according as the volume integrals extend throughout the space external or internal to the sphere.

Putting $\phi = \chi$, $\psi = r^{-1}$, and applying the theorem to the space *outside* the sphere $r = at$, we obtain

$$-\iint \frac{\chi}{r^2}\, dS = \iint \frac{1}{r}\frac{d\chi}{dr}\, dS + \iiint \frac{1}{r} \nabla^2 \chi\, d\alpha d\beta d\gamma \;\ldots\ldots(13).$$

Putting $\phi = \chi$, $\psi = 1$, and applying the theorem to the space *inside* the sphere $r = at$, we obtain

$$\iiint \nabla^2 \chi\, d\alpha d\beta d\gamma = \iint \frac{d\chi}{dr}\, dS \;\ldots\ldots\ldots\ldots\ldots(14).$$

Eliminating $d\chi/dr$ between (13) and (14), we obtain

$$-\frac{1}{4\pi a}\iiint \nabla^2 \chi\, d\alpha d\beta d\gamma \quad (r < at) - \frac{t}{4\pi}\iiint \frac{1}{r} \nabla^2 \chi\, d\alpha d\beta d\gamma \;(r > at)$$
$$= \frac{t}{4\pi}\iint \frac{\chi}{r^2}\, dS = \frac{t}{4\pi}\frac{d}{dx}\iint F(at)\, d\Omega \ldots(15),$$

by (11) and (12).

If $\dot{u}_0$, $\dot{v}_0$, $\dot{w}_0$ be the initial velocities,

$$\frac{d}{dx}\left(\frac{d\dot{u}_0}{dx} + \frac{d\dot{v}_0}{dy} + \frac{d\dot{w}_0}{dz}\right) = \frac{d}{dx}\left(\frac{d^2F}{dx^2} + \frac{d^2F}{dy^2} + \frac{d^2F}{dz^2}\right) = \nabla^2\chi \ldots(16).$$

Now the function χ, and consequently the functions $\dot{u}_0$, $\dot{v}_0$, $\dot{w}_0$, when they occur in a triple integral, are functions of the position of the point whose coordinates are $x + \alpha$, $y + \beta$, $z + \gamma$; whence $d/dx = d/d\alpha$, and accordingly we may write $d/d\alpha$, &c. for d/dx, &c. Hence substituting the value of χ from (16) in (15), integrating by parts, and observing that the two surface integrals which appear in the integration cancel one another, we obtain

$$\frac{t}{4\pi}\frac{d}{dx}\iint F(at)\, d\Omega = -\frac{t}{4\pi}\iiint \left(\alpha \frac{d\dot{u}_0}{d\alpha} + \beta \frac{d\dot{v}_0}{d\alpha} + \gamma \frac{d\dot{w}_0}{d\alpha}\right) r^{-3}\, d\alpha d\beta d\gamma,$$
$$r > at.$$

Integrating the right-hand side again by parts, it follows that

if u_1' be the portion of u which depends upon the initial velocity of dilatation,

$$u_1' = \frac{t}{4\pi r^3}\iint l\,(\alpha\dot{u}_0 + \beta\dot{v}_0 + \gamma\dot{w}_0)\,dS$$
$$+ \frac{t}{4\pi}\iiint\left(\dot{u}_0\frac{d}{d\alpha}\frac{\alpha}{r^3} + \dot{v}_0\frac{d}{d\alpha}\frac{\beta}{r^3} + \dot{w}_0\frac{d}{d\alpha}\frac{\gamma}{r^3}\right)d\alpha d\beta d\gamma.$$

Let q_0 be the initial velocity along OP, so that

$$q_0 = l\dot{u}_0 + m\dot{v}_0 + n\dot{w}_0,$$

and let $(q_0)_{at}$ denote the value of q_0 at a distance at from O, then the surface integral

$$= \frac{t}{4\pi r^2}\iint l\,(q_0)_{at}\,dS = \frac{t}{4\pi}\iint l\,(q_0)_{at}\,d\Omega\,;$$

also the triple integral can easily be shown to be equal to

$$\iiint(\dot{u}_0 - 3lq_0)\,r^{-3}\,d\alpha d\beta d\gamma\,;$$

whence we finally obtain for the portion of u depending upon the initial rate of dilatation

$$u_1' = \frac{t}{4\pi}\iint l\,(q_0)_{at}\,d\Omega$$
$$+ \frac{t}{4\pi}\iiint(\dot{u}_0 - 3lq_0)\,r^{-3}\,d\alpha d\beta d\gamma,\ (r > at)\ldots\ldots(17).$$

222. We must now find the portion of u due to the initial velocities of rotation.

Applying Green's theorem to the space outside the sphere $r = bt$, by writing F_1 for χ in (13), we obtain

$$\iint F_1 d\Omega = \iint \frac{F_1}{r^2}\,dS$$
$$= -\iint\frac{1}{r}\frac{dF_1}{dr}\,dS - \iiint r^{-1}\nabla^2 F_1\,d\alpha d\beta d\gamma,\ (r > bt)\ \ldots(18).$$

Since $$\frac{dF_1}{d\alpha} + \frac{dF_2}{d\beta} + \frac{dF_3}{d\gamma} = 0,$$

by (3), we have

$$0 = \frac{1}{r}\iiint\frac{d}{d\alpha}\left(\frac{dF_1}{d\alpha} + \frac{dF_2}{d\beta} + \frac{dF_3}{d\gamma}\right)d\alpha d\beta d\gamma$$
$$= \frac{1}{r}\iint\left(l\frac{dF_1}{d\alpha} + m\frac{dF_2}{d\alpha} + n\frac{dF_3}{d\alpha}\right)dS.$$

Adding this to (18), we obtain

$$\iint F_1 d\Omega = \iint \frac{1}{r}\left\{m\left(\frac{dF_2}{d\alpha} - \frac{dF_1}{d\beta}\right) - n\left(\frac{dF_1}{d\gamma} - \frac{dF_3}{d\alpha}\right)\right\} dS$$
$$- \iint r^{-1}\nabla^2 F_1 d\alpha d\beta d\gamma, \ (r > bt) \ldots\ldots(19).$$

Now if $\dot{\xi}_0$, $\dot{\eta}_0$, $\dot{\zeta}_0$ be the initial velocities of rotation,

$$\frac{dF_2}{d\alpha} - \frac{dF_1}{d\beta} = \dot{\zeta}_0 = \frac{d\dot{v}_0}{d\alpha} - \frac{d\dot{u}_0}{d\beta},$$

and
$$\nabla^2 F_1 = \frac{d\dot{\eta}_0}{d\gamma} - \frac{d\dot{\zeta}_0}{d\beta} = \nabla^2 \dot{u}_0;$$

whence (19) becomes

$$\iint F_1 d\Omega = \iint r^{-1}(m\dot{\zeta}_0 - n\dot{\eta}_0)\, dS - \iiint r^{-1}\left(\frac{d\dot{\eta}_0}{d\gamma} - \frac{d\dot{\zeta}_0}{d\beta}\right) d\alpha d\beta d\gamma$$
$$= \iiint (\beta\dot{\zeta}_0 - \gamma\dot{\eta}_0)\, r^{-3} d\alpha d\beta d\gamma$$
$$\alpha\frac{d\dot{u}_0}{d\alpha} + \beta\frac{d\dot{v}_0}{d\alpha} + \gamma\frac{d\dot{w}_0}{d\alpha}\Big)\, r^{-3} d\alpha d\beta d\gamma$$
$$\alpha\frac{d\dot{u}_0}{d\alpha} + \beta\frac{d\dot{u}_0}{d\beta} + \gamma\frac{d\dot{u}_0}{d\gamma}\Big)\, r^{-3} d\alpha d\beta d\gamma.$$

From the last article it follows, that the first triple integral

$$= -\iint l(q_0)_{bt}\, d\Omega - \iiint (\dot{u}_0 - 3lq_0)\, r^{-3} d\alpha d\beta\gamma, \ (r > bt).$$

Since
$$d\alpha d\beta d\gamma = drdS = r^2 drd\Omega,$$

the second triple integral

$$= -\iiint \frac{d\dot{u}_0}{dr} drd\Omega = \iint (\dot{u}_0)_{bt}\, d\Omega;$$

whence the portion of u, which depends on the initial velocity of rotation, is

$$u_2' = \frac{t}{4\pi}\iint (\dot{u}_0 - lq_0)_{bt}\, d\Omega - \frac{t}{4\pi}\iiint (\dot{u}_0 - 3lq_0)\, r^{-3} d\alpha d\beta d\gamma,$$
$$(r > bt) \ldots\ldots\ldots\ldots(20).$$

To obtain the portion of u due to the initial velocities, we must add the right-hand sides of (17) and (20), and must recollect, that in (17) the limits of r are ∞ and at, and in (20) the limits are ∞ and bt; we thus finally obtain

$$u' = \frac{t}{4\pi}\iint l(q_0)_{at}\, d\Omega + \frac{t}{4\pi}\iint (\dot{u}_0 - lq_0)_{bt}\, d\Omega$$
$$+ \frac{t}{4\pi}\iiint (3lq_0 - \dot{u}_0)\, r^{-3} d\alpha d\beta d\gamma, \ (bt < r < at) \ldots\ldots(21).$$

This is the portion of u which depends upon the initial velocities, expressed in terms of these quantities.

223. In order to obtain the portion of u due to the initial displacements u_0, v_0, w_0, we must change $\dot{u}_0$, $\dot{v}_0$, $\dot{w}_0$, q_0 into u_0, v_0, w_0, ρ_0, where

$$\rho_0 = lu_0 + mv_0 + nw_0,$$

and differentiate with respect to t. Now u_0 is some function of the position of a point; hence $(u_0)_{at}$, which is the value of u_0 at some point on the surface of a sphere whose centre is O and radius at, is some function of

$$x + lat, \quad y + mat, \quad z + nat.$$

It therefore follows that

$$\begin{aligned}\frac{d}{dt}(u_0)_{at} &= a\left(l\frac{d}{dx} + m\frac{d}{dy} + n\frac{d}{dz}\right)(u_0)_{at}\\ &= a\frac{d}{dr}(u_0)_{at}.\end{aligned}$$

The triple integral is taken throughout the space bounded by the two spheres whose common centre is O, and whose radii are respectively equal to at and bt. If therefore we write $r^2drd\Omega$ for an element of volume, the triple integral may be written

$$\iint\int_{bt}^{at} (3l\rho_0 - u_0)\, r^{-1}\, drd\Omega\,;$$

and therefore its differential coefficient with respect to t, is

$$t^{-1}\iint(3l\rho_0 - u_0)_{at}\, d\Omega - t^{-1}\iint(3l\rho_0 - u_0)_{bt}\, d\Omega.$$

Hence the portion of u which depends upon the initial displacements is

$$\begin{aligned}u'' = &\frac{1}{4\pi}\iint\left(4l\rho_0 + lat\frac{d\rho_0}{dr} - u_0\right)_{at} d\Omega\\ &+ \frac{1}{4\pi}\iint\left(2u_0 + bt\frac{du_0}{dr} - 4l\rho_0 - lbt\frac{d\rho_0}{dr}\right)_{bt} d\Omega\\ &+ \frac{1}{4\pi}\iiint(3l\rho_0 - u_0)\, r^{-3}\, d\alpha d\beta d\gamma, \quad (bt < r < at) \quad\text{......(22).}\end{aligned}$$

The complete value of the displacement u, due to the initial displacements and velocities, is obtained by adding the values of u', u'' given by (21) and (22). The values of v and w can be written down from symmetry.

224. Sir G. Stokes has applied these results to the solution of two important problems, viz. (i) the determination of the disturbance produced by a given variable force acting in a given direction at a given point of the medium; (ii) the determination of the law of disturbance in a secondary wave of light.

We shall now proceed to consider the first problem.

Disturbance produced by a given Force.

225. Let P be the point at which the force acts; and let T be a small space described about P, which will ultimately be supposed to vanish, and let O be a point outside T at which the value of the disturbance is sought; also let D be the density of the medium.

Let t be the time of observation, measured from some previous epoch; and let t' be the time, which the dilatational wave occupies in travelling from P to O.

Let $f(t)$ be the given force, and $F(t)$ the velocity at P produced by the force during a very small interval of time dt', then the usual equation of motion gives

$$DT\frac{dv}{dt}=f(t).$$

Now if we consider the state of things which was going on at P at a time t' ago, we must in this equation write $t-t'$ for t, and dt' for dt; also $\delta v=F(t-t')$, whence

$$F(t-t')=\frac{f(t-t')\,\delta t'}{DT} \quad\text{.................}(23).$$

This equation gives the value of the velocity communicated during the interval $\delta t'$ in terms of the force.

Let O be the origin, $OP=r$; also let l, m, n be the direction cosines of OP, and l', m', n' those of the force; and let k be the angle between OP and the direction of the force, so that

$$k=ll'+mm'+nn'.$$

Since the disturbance may by virtue of (23) be regarded as one which is produced by a given initial velocity, the resulting disturbance at O is determined by (21); also since

$$q_0=kF,$$

it follows that the first term of (21) becomes

$$\frac{t'}{4\pi DT}\iint lkf(t-t')\,\delta t' d\Omega = \frac{1}{4\pi a^2 DT}\iint lkf\left(t-\frac{r}{a}\right) r^{-1}\,\delta r dS,$$

since $r = at'$.

Since the force is supposed to have commenced to act an infinitely long time ago, we must integrate this expression with respect to t' between the limits r/a and $-\infty$; but since the force is confined to the indefinitely small volume T, $f(t-t')$ will be insensible except for values of t' comprised between the narrow limits r_1/a and r_2/a, where r_1, r_2 are the least and greatest values of the radius vector drawn from O to T. We may therefore omit the integral signs, and replace $\delta r dS$ by T, and we thus obtain for the value of the first term of (21),

$$\frac{lk}{4\pi Da^2 r} f\left(t-\frac{r}{a}\right) \quad\text{.....................(24).}$$

If we denote by t'', the time which a distortional wave occupies in travelling from P to O, and treat the second term of (21) in a similar manner, we shall obtain

$$\frac{l'-lk}{4\pi Db^2 r} f\left(t-\frac{r}{b}\right) \quad\text{.....................(25).}$$

In order to find what the triple integral in (21) becomes, we see from (17) and (20) that it may be written

$$\frac{t'}{4\pi}\iiint (\dot{u}_0 - 3lq_0)\, r^{-3}\, d\alpha d\beta d\gamma \quad (r > at')$$

$$-\frac{t''}{4\pi}\iiint (\dot{u}_0 - 3lq_0)\, r^{-3}\, d\alpha d\beta d\gamma \quad (r > bt'').$$

The first term of this accordingly becomes

$$\frac{t'}{4\pi DT}\iiint (l'-3lk) f(t-t')\, dt' r^{-3}\, d\alpha d\beta d\gamma.$$

Since $f(t-t')$ is insensible except throughout the space T, we may write T for $d\alpha d\beta d\gamma$, and omit the integral signs; we thus obtain

$$\frac{t'}{4\pi Dr^3}(l'-3lk) f(t-t')\, dt',$$

and this has to be integrated with respect to t' between the limits r/a and $-\infty$. This term thus becomes

$$\frac{l'-3lk}{4\pi Dr^3}\int_{-\infty}^{\frac{r}{a}} t' f(t-t')\, dt'.$$

Treating the second term in the same manner, and remembering that the limits of t'' are r/b and $-\infty$, and adding we obtain

$$\frac{3lk - l'}{4\pi D r^3} \int_{\frac{r}{a}}^{\frac{r}{b}} t' f(t - t')\, dt'.$$

Hence collecting all the terms we obtain

$$u = \frac{lk}{4\pi D a^2 r} f\left(t - \frac{r}{a}\right) + \frac{l' - lk}{4\pi D b^2 r} f\left(t - \frac{r}{b}\right)$$
$$+ \frac{3lk - l'}{4\pi D r^3} \int_{\frac{r}{a}}^{\frac{r}{b}} t' f(t - t')\, dt' \quad \ldots\ldots\ldots\ldots\ldots(26).$$

The values of v and w are obtained by putting m, m'; n, n' respectively for l, l'. If therefore we take OP for the axis of x, and the plane passing through OP and the direction of the force as the plane xz, and put ϕ for the inclination of the force to PO, we shall have

$$l = 1,\ m = 0,\ n = 0; \quad l' = k = \cos\phi,\ m' = 0,\ n' = \sin\phi.$$

Whence

$$\left.\begin{aligned} u &= \frac{\cos\phi}{4\pi D a^2 r} f\left(t - \frac{r}{a}\right) + \frac{\cos\phi}{2\pi D r^3} \int_{r}^{\frac{r}{b}} t' f(t - t')\, dt' \\ v &= 0 \\ w &= \frac{\sin\phi}{4\pi D b^2 r} f\left(t - \frac{r}{b}\right) - \frac{\sin\phi}{4\pi D r^3} \int_{r}^{b} t' f(t - t')\, dt' \end{aligned}\right\} \ldots(27).$$

226. In discussing this result Sir G. Stokes says:

"The first term in u represents a disturbance which is propagated from P with a velocity a. Since there is no corresponding term in v or w, the displacement, as far as relates to this disturbance, is strictly normal to the front of the wave. The first term in w represents a disturbance which is propagated from P with a velocity b, and as far as relates to this disturbance, the displacement takes place strictly in the front of the wave. The remaining terms in u and w represent a disturbance of the same kind as that which takes place in an incompressible fluid, in consequence of the motion of solid bodies in it. If $f(t)$ represent a force which acts for a short time, and then ceases, $f(t - t')$ will

differ from zero only between certain narrow limits of t, and the integral contained in the last terms of u and w will be of the order r, and therefore the terms themselves will be of the order r^{-2}, whereas the leading terms are of the order r^{-1}. Hence in this case the former terms will not be sensible beyond the immediate neighbourhood of P. The same will be true if $f(t)$ represent a periodic force, the mean value of which is zero. But if $f(t)$ represent a force always acting one way, as for example a constant force, the last terms in u and w will be of the same order, when r is large, as the first terms.

"It has been remarked, that there is strong reason for believing that in the case of the luminiferous ether, the ratio of a/b is extremely large if not infinite. Consequently the first term of u, which relates to normal vibrations, will be insensible, if not absolutely evanescent. In fact, if the ratio a/b were no greater than 100, the denominator in this term would be 10000 times as great as the denominator of the first term of w. Now the molecules of a solid or gas in the act of combustion are probably thrown into a state of violent vibration, and may be regarded, at least very approximately, as centres of disturbing forces. We may thus see why transversal vibrations should be alone produced, unaccompanied by normal vibrations, or at least by any which are of sufficient magnitude to be sensible. If we could be sure that the ether was strictly incompressible, we should of course be justified in asserting that normal vibrations are impossible.

"If we suppose that $a=\infty$, and $f(t)=F\sin 2\pi bt/\lambda$, we shall obtain from (27)

$$\left.\begin{aligned}
u &= \frac{F\lambda\cos\phi}{4\pi^2Db^2r^2}\cos\frac{2\pi}{\lambda}(bt-r)-\frac{F\lambda^2\cos\phi}{4\pi^3Db^2r^3}\sin\frac{\pi r}{\lambda}\cos\frac{2\pi}{\lambda}(bt-\tfrac{1}{2}r)\\
v &= 0\\
w &= \frac{F\sin\phi}{4\pi Db^2r}\sin\frac{2\pi}{\lambda}(bt-r)-\frac{F\lambda\sin\phi}{8\pi^2Db^2r^2}\cos\frac{2\pi}{\lambda}(bt-r)\\
&\quad+\frac{F\lambda^2\sin\phi}{8\pi^3Db^2r^3}\sin\frac{\pi r}{\lambda}\cos\frac{2\pi}{\lambda}(bt-\tfrac{1}{2}r)
\end{aligned}\right\}\ldots(28),$$

and we see that the most important term of u is of the order $\lambda/\pi r$ compared with the leading term of w, which represents transversal vibrations properly so called. Hence u and the second and third terms of w, will be insensible, except at a distance from P comparable with λ, and may be neglected; but the existence of

terms of this nature, in the case of a spherical wave whose radius is not regarded as infinite, must be borne in mind, in order to understand in what manner transversal vibrations are compatible with the absence of dilatation or condensation."

Determination of the Law of Disturbance in a Secondary Wave of Light.

227. Let us suppose, that plane waves of light are travelling through an elastic medium. Let the axis of x be parallel to the direction of propagation of the waves, whilst the axis of z is parallel to the direction of vibration; then the displacement at any point of the medium may be denoted by

$$w = f(bt - x).$$

Let P be a fixed point, which we shall choose as the origin; O a point whose coordinates are x, y, z; dS a small element of the plane yz, which contains P. We require to find that portion of the total disturbance at O, which is due to the element dS at P.

The disturbance at dS consists of a displacement $f(bt)$ and a velocity $bf'(bt)$. In order to find the disturbance at O due to the velocity, let t' be the time which the disturbance occupies in travelling from P to O; then if $PO = r$, we shall have $r = bt'$; also let l, m, n be the direction cosines of OP measured from O, so that $-l, -m, -n$ are the direction cosines of OP measured from P.

We shall thus have

$$(q_0)_{bt} = -nbf'(bt - bt');$$

also since the dilatational terms are to be omitted on account of the largeness of a, the displacement corresponding to that part of the disturbance which is due to the velocity, which existed at P at time t' ago, is given by (20). Since the volume integral varies as r^{-3}, it must be omitted; whence recollecting that the signs of l, m, n in (22) must be reversed, we obtain for the portion depending on dS,

$$u = -\frac{lnbt'}{4\pi} f'(bt - bt')\, d\Omega.$$

In order to find $d\Omega$ in terms of dS, let us consider a thin film comprised between dS and a parallel surface, whose thickness is bdt'. Then the volume of this slice is $bdt'dS$; but this volume is

also equal to $r^2 d\Omega dr$; and since $r = bt'$, it follows that $r^2 d\Omega = dS$; whence

$$u = -\frac{lndS}{4\pi r} f'(bt - r) \text{ ....(29).}$$

Treating v and w in a similar manner, we obtain

$$\left.\begin{aligned} v &= -\frac{mndS}{4\pi r} f'(bt - r) \\ w &= \frac{(1 - n^2)\, dS}{4\pi r} f'(bt - r) \end{aligned}\right\} \text{ ....(30).}$$

Equations (29) and (30) show that $lu + mv + nw = 0$, from which we see that the displacement takes place in a plane through O, perpendicular to PO; also since $u/v = l/m$, it takes place in a plane through PO and the axis of z, which is the direction of vibration of the primary wave. Putting $n = \cos\phi$, so that ϕ is the angle between PO and the axis of z, the magnitude of this displacement is

$$\zeta_1 = \frac{dS}{4\pi r} \sin\phi f'(bt - r) \text{ ....(31).}$$

228. The portion of the displacement at O, which depends on the initial displacement at P, can be obtained in a similar manner from (22). Since we neglect all terms of a higher order than r^{-1}, the only terms of (22) which it will be necessary to retain, are those which involve the differential coefficients of u_0, v_0, w_0, ρ_0 in the second double integral. Writing $r = bt'$, $dS = r^2 d\Omega$, the value of u'' becomes

$$\frac{ldS}{4\pi r}\left(\frac{d\rho_0}{dr}\right)_{bt'},$$

since the signs of l, m, n in (22) have to be changed.

In order to determine this differential coefficient, let x', y', z' be the coordinates of P referred to any origin, then

$$\rho_0 = -nw_0 = -nf(bt - bt' - x'),$$

and
$$\frac{d\rho_0}{dr} = -l\frac{d\rho_0}{dx'} = -lnf'(bt - bt' - x');$$

where the accent in f' denotes differentiation with respect to bt. Transferring the origin to P, and recollecting that $bt' = r$, we obtain

$$\left(\frac{d\rho_0}{dr}\right)_{bt'} = -lnf'(bt - r),$$

whence
$$u = -\frac{l^2 ndS}{4\pi r} f'(bt - r) \text{ ....(32).}$$

Treating v and w in the same manner, we obtain

$$\left.\begin{aligned} v &= -\frac{lmn\,dS}{4\pi r} f'(bt-r) \\ w &= \frac{l(1-n^2)\,dS}{4\pi r} f'(bt-r) \end{aligned}\right\} \qquad \text{(33)}.$$

This is the portion of the displacement at O, which depends upon the displacement at P. If we denote it by ζ_2, and put $l = \cos\theta$, we see that its direction is the same as that of ζ_1, and its magnitude is

$$\zeta_2 = \frac{dS}{4\pi r}\cos\theta\sin\phi f'(bt-r) \qquad \text{(34)}.$$

229. By combining the results of (31) and (34) we obtain the important theorem, which was enunciated in § 37.

Let $u=0$, $v=0$, $w=f(bt-x)$ be the displacements corresponding to the primary wave; let P be any point in the plane yz, dS an element of that plane adjacent to P; and consider the disturbance due to that portion only of the incident disturbance, which passes continually across dS. Let O be any point of the medium situated at a distance from P, which is large in comparison with the wave-length of light; let $PO=r$, and let this line respectively make angles θ and ϕ with the direction of propagation of the incident light, and with the direction of vibration. Then the displacement at O will take place in a direction perpendicular to PO and lying in the plane zPO, and if ζ be the displacement at O reckoned positive in the direction nearest to that in which the incident vibrations are reckoned positive,

$$\zeta = \frac{dS}{4\pi r}(1+\cos\theta)\sin\phi f'(bt-r) \qquad \text{(35)}.$$

In particular if

$$f(bt-x) = c\sin\frac{2\pi}{\lambda}(bt-x)$$

we shall have

$$\zeta = \frac{c\,dS}{2\lambda r}(1+\cos\theta)\sin\phi\cos\frac{2\pi}{\lambda}(bt-r) \qquad \text{(36)}.$$

This equation, as was stated in § 37, determines the law of disturbance in the secondary wave proceeding from the element dS of the primary wave.

Sir G. Stokes has verified this result, by showing that if the right-hand side of (36) be integrated over the whole area of the plane yz, the result will be

$$\zeta = c \sin \frac{2\pi}{\lambda}(bt - x), \text{ or } 0$$

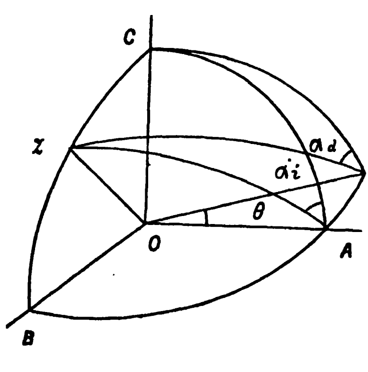
according as x is positive or negative.

Hence the disturbance, continually transmitted across the plane yz, produces the same disturbance in front of that plane, as if the wave had not been broken up, and does not produce any back wave.

The whole of the preceding results are of a purely mathematical character, and are therefore applicable to any medium whatever, whose motion is capable of being represented by equations of the same form as (1).

Vibrations of Polarized Light are Perpendicular to the Plane of Polarization.

230. We must now, following Stokes, explain how the preceding results are employed to determine, whether the vibrations of polarized light are in or perpendicular to the plane of polarization.



In the figure, let the incident wave, which will be supposed to be plane polarized, be parallel to the plane OBC, and let OA be perpendicular to OBC, so that OA is the direction of propagation of the incident light; also let its direction of vibration be parallel to OZ.

Let OD be any diffracted ray lying in the plane OBA, then equation (36) shows, that the direction of vibration of the diffracted ray OD lies in the plane ZOD.

We shall call the plane which passes through a ray and contains the direction of vibration, the *plane of vibration;* we shall also call the angle between the incident ray produced and the diffracted ray, the *angle of diffraction;* and the plane containing these two rays the *plane of diffraction.* Whence OZA and OZD are the planes of vibration of the incident and diffracted rays; AOD is the angle of diffraction, and the plane $ODAB$ is the plane of diffraction.

Let θ be the angle of diffraction; and let α_i, α_d be the angles, which the planes of vibration of the incident and diffracted rays respectively make with planes drawn through those rays perpendicularly to the plane of diffraction. Then

$$AOD = \theta, \quad ZAC = \alpha_i, \quad ZDC = \alpha_d.$$

From the spherical triangle DZB, we obtain

$$\sin DB = \tan BZ \cot ZDB,$$

or

$$\tan \alpha_d = \cos \theta \tan \alpha_i \dots\dots\dots\dots\dots\dots \quad (37).$$

If the plane of vibration ZAO of the incident ray, be made to turn round OA with uniform angular velocity ω, it follows that the plane of vibration ZDO of the diffracted ray will turn round with variable velocity. In order to see this, differentiate (37) with respect to t; then since θ is constant, and $d\alpha_i/dt = \omega$, we obtain

$$\frac{d\alpha_d}{dt} = \frac{\omega \cos \theta}{1 - \sin^2\theta \sin^2\omega t}.$$

From this equation we observe, that as t increases from 0 to $\pi/2\omega$, $d\alpha_d/dt$ increases in value. It therefore follows that as α_i increases, the planes of vibration of the diffracted rays will not be distributed uniformly, but will be crowded *towards the plane perpendicular to the plane of diffraction*, according to the law expressed by the above equation.

Now the angles which the planes of polarization of the incident and diffracted rays, (if the diffracted rays prove to be really plane polarized), make with planes perpendicular to the plane of diffraction, can be measured by means of a pair of graduated instruments furnished with Nicol's prisms; and the readings of

the instrument, which is used as the analyser, will show whether the planes of polarization of the diffracted rays are crowded towards the plane of diffraction, or towards the plane perpendicular to the plane of diffraction.

Let ϖ, α be the azimuths of the planes of polarization of the incident and diffracted rays, both measured from planes perpendicular to the plane of diffraction. Then if the vibrations of polarized light are *in* the plane of polarization, the planes ZOA and ZOD will respectively be the planes of polarization of the incident and diffracted rays; accordingly on this hypothesis we should have $\varpi = \alpha_i$, $\alpha = \alpha_d$ and therefore by (37)

$$\tan \alpha = \cos \theta \tan \varpi;$$

and we have already shown that in this case, the planes of polarization will be crowded *towards the plane perpendicular to the plane of diffraction*. But if the vibrations of polarized light are *perpendicular* to the plane of polarization, we shall have $\varpi = \frac{1}{2}\pi + \alpha_i$, $\alpha = \frac{1}{2}\pi + \alpha_d$, in which case

$$\tan \alpha = \sec \theta \tan \varpi;$$

and from this equation it follows, that the planes of polarization of the diffracted rays will be crowded *towards the plane of diffraction*. We have thus a crucial test for deciding, whether the vibrations of polarized light are in or perpendicular to the plane of polarization.

An elaborate series of experiments was made by Sir G. Stokes, which are described in the paper from which this investigation is taken, and he found that the planes of polarization of the diffracted rays were very decidedly crowded *towards* the plane of diffraction; whence the results of the experiments confirm Fresnel's hypothesis, that the vibrations of polarized light are perpendicular to the plane of polarization.

Resolution of Plane Waves.

231. The question of the resolution of plane waves has recently been discussed by Lord Rayleigh[1], who has argued that the method of resolution employed by Stokes is the simplest, but is only one out of an indefinite number which might be proposed, all of which are equally legitimate so long as the question is regarded as a purely mathematical one; and that the purely mathematical question has no definite answer. For the purpose of investigating this point, I have recently discussed the corresponding case of plane waves of sound, and have shown[2], that the most general expression for the disturbance produced by an element dS of the plane wave of sound, whose velocity potential is

$$\phi = \cos \frac{2\pi}{\lambda}(at - x)$$

is the real part of

$$\frac{\iota\kappa\epsilon^{\iota\kappa at} dS}{4\pi(2n+1)}\{n\psi_{n-1} P_{n-1} + (2n+1)\psi_n P_n + (n+1)\psi_{n+1} P_{n+1}\} \ldots (38),$$

where n is zero or any positive integer, P_n is a zonal harmonic, $k = 2\pi/\lambda$, and

$$\psi_n = \frac{\epsilon^{-\iota\kappa r}}{r} f_n(\iota\kappa r) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (39),$$

where

$$f_n(x) = 1 + \frac{n(n+1)}{2 \,.\, x} + \frac{(n-1)n(n+1)(n+2)}{2 \,.\, 4 \,.\, x^2} + \ldots\ldots\ldots$$
$$+ \frac{1 \,.\, 2 \,.\, 3 \ldots\ldots 2n}{2 \,.\, 4 \,.\, 6 \ldots\ldots 2n \,.\, x^n} \ldots\ldots\ldots\ldots\ldots (40).$$

Since the function $\psi_n P_n$ is the velocity potential of a multiple source of sound of the nth order, it follows that the effect of the element may be represented by three multiple sources of orders $n-1$, n, $n+1$. If in (38) we put $n = 0$, and realize, the result is

$$-\frac{dS}{2\lambda r}(1 + \cos\theta)\sin\frac{2\pi}{\lambda}(at - r) + \frac{dS\cos\theta}{4\pi r^2}\cos\frac{2\pi}{\lambda}(at - r),$$

where r is the distance of the element from a point P, and θ is the angle which the direction of r makes with that of propagation.

If λ is small compared with r, as is always the case in optical problems, the first term is the most important.

[1] *Encycl. Brit.* Art. "Wave Theory," pp. 452–454.
[2] *Proc. Lond. Math. Soc.* vol. XXII. p. 317.

232. From the preceding result, it might be anticipated, that Stokes' formula is equivalent to the combination of a simple and a double source of light; and we shall now show that this is the case.

We have shown in § 215, equation (6), that the most general solution of the equations of motion of an elastic solid is given by the equations

$$u = d\phi/dx + u'\epsilon^{\iota\kappa bt}, \quad v = d\phi/dy + v'\epsilon^{\iota\kappa bt}, \quad w = d\phi/dz + w'\epsilon^{\iota\kappa bt},$$

where ϕ is the function which determines the longitudinal wave; and u', v', w' each satisfy an equation of the form

$$(\nabla^2 + \kappa^2)\, u' = 0 \quad \text{..........................(41)},$$

subject to the condition

$$\frac{du'}{dx} + \frac{dv'}{dy} + \frac{dw'}{dz} = 0 \quad \text{..................(42)}.$$

It is well known[1], that (41) are satisfied by

$$u' = r^{-n}\psi_n X_n, \quad v' = r^{-n}\psi_n Y_n, \quad w' = r^{-n}\psi_n Z_n \text{.........(43)}$$

where X_n, Y_n, Z_n are three solid harmonics of positive degree n, and ψ_n is the function defined by (39) and (40). The function ψ can also be shown to satisfy the following equations, viz.

$$\left.\begin{aligned} r\frac{d\psi_n}{dr} - n\psi_n &= -\iota\kappa r\psi_{n+1} \\ r\frac{d\psi_n}{dr} + (n+1)\,\psi_n &= -\iota\kappa r\psi_{n-1} \\ (2n+1)\,\psi_n &= \iota\kappa r\,(\psi_{n+1} - \psi_{n-1}) \end{aligned}\right\} \text{...(44)},$$

which are frequently useful.

Let ϕ_n, χ_n, be any positive solid harmonics of degree n; then by means of a process similar to that employed by Lamb[2], it can be shown that

$$u' = \frac{\psi_n}{r^n}\left(\frac{d\phi_{n+1}}{dx} + y\frac{d\chi_n}{dz} - z\frac{d\chi_n}{dy}\right) + \frac{(n+1)\,\psi_{n+2}r^{2n+3}}{n+2}\frac{d}{dx}\left(\frac{\phi_{n+1}}{r^{2n+3}}\right)$$
$$\text{........................(45)}$$

with symmetrical expressions for v', w'.

The simplest way of verifying this result is to recollect, that a solid harmonic is a homogeneous function of x, y, z of degree n;

[1] Lord Rayleigh, *Theory of Sound*, Ch. XVII.; Stokes, On the communication of vibrations from a vibrating body to the atmosphere, *Phil. Trans.* 1868.

[2] *Proc. Lond. Math. Soc.* vol. XIII. p. 51; see also, Basset, *Hydrodynamics*, vol. II. pp. 316—318.

we thus see that the first term of u' is of the form $r^{-n}\psi_n X_n$, whilst the second is of the form $r^{-n-2}\psi_{n+2}X_{n+2}$. The above expression therefore satisfies (41). Also if we differentiate the expressions for u', v', w' with respect to x, y, z, and take account of (44), it will be found that (42) is satisfied.

233. The simplest solution is obtained by putting $n=0$, in which case

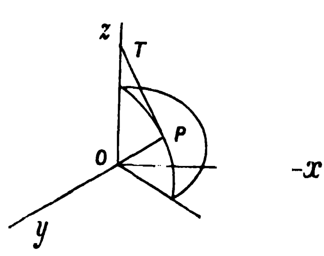
$$\psi_0 = r^{-1}\epsilon^{-\iota\kappa r}, \quad \phi_1 = Ax + By + Cz, \quad \chi_0 = \text{const.}$$

whence

$$u_0 = \frac{A\epsilon^{-\iota\kappa r}}{2r^3}\left\{1 + \frac{3}{\iota\kappa r} + \frac{3}{(\iota\kappa r)^2}\right\}\left\{Ar^2 - 3(Ax + By + Cz)x\right\} \quad \text{...............(46).}$$

This expression may be regarded as giving the value of u for a *simple source of light*, and it corresponds to a source in hydrodynamics, or to an electrified point. The expression is, however, more analogous to a doublet or a magnet, inasmuch as a simple source of light has *direction* as well as *magnitude*. The direction cosines of the axis of the source are proportional to A, B, C; and, if we suppose that its axis is parallel to z, we shall have $A = B = 0$. Also, in optical problems, λ is usually so small in comparison with r, that at a considerable distance from the source, powers of $(\kappa r)^{-1}$ may be neglected; whence, writing $F = \frac{3}{2}C$, we shall obtain

$$\left.\begin{aligned} u_0 &= -\frac{Fxz}{r^3}\epsilon^{-\iota\kappa r} \\ v_0 &= -\frac{Fyz}{r^3}\epsilon^{-\iota\kappa r} \\ w_0 &= \frac{F(x^2+y^2)}{r^3}\epsilon^{-\iota\kappa r} \end{aligned}\right\} \quad \text{.(47).}$$



In the figure, let P be the point x, y, z;

also let $\theta = POx$, $\phi = POz$;

draw PT perpendicular to OP in the plane POz. Then the preceding equations show, that the direction of vibration is along PT, and its magnitude is equal to

$$\frac{F}{r}\epsilon^{-\iota\kappa r}\sin\phi.$$

Restoring the time factor $\epsilon^{\iota\kappa bt}$ and realizing, this becomes

$$\frac{F}{r}\sin\phi\cos\frac{2\pi}{\lambda}(bt-r) \quad\text{..................(48).}$$

This is the expression for the disturbance produced by a simple source of light at a point r, whose distance from the source is large compared with the wave-length.

The motion is, as might be expected, symmetrical with respect to the axis of z, and vanishes on that axis where $\phi=0$ or π; and it is a maximum on the plane xy where $\phi=\frac{1}{2}\pi$.

234. In order to obtain the most general expression for a singular point of the second order, we must put $n=1$; whence

$$\phi_2=(B-C)x^2+(C-A)y^2+(A-B)z^2+2A'yz+2B'zx+2C'xy,$$

$$\chi_1=\alpha x+\beta y+\gamma z;$$

and

$$u_1=\frac{\psi_1}{r}\left(\frac{d\phi_2}{dx}+\gamma y-\beta z\right)+\tfrac{2}{3}\psi_3 r^4\frac{d}{dx}\left(\frac{\phi_2}{r^5}\right) \quad\text{..................(49).}$$

The expression for a singular point of the second order accordingly contains eight constants, and is therefore a function of considerable generality. Let us now suppose, as a particular case, that

$$\phi_2=\tfrac{3}{10}Fzx,\quad \chi_1=\tfrac{1}{2}Fy;$$

then, if we confine our attention to points at a considerable distance from the origin, we may put

$$\psi_1=\psi_3=r^{-1}\epsilon^{-\iota\kappa r},$$

whence

$$\left.\begin{aligned} u_1&=-\frac{Fx^2z}{r^4}\epsilon^{-\iota\kappa r}\\ v_1&=-\frac{Fxyz}{r^4}\epsilon^{-\iota\kappa r}\\ w_1&=\frac{Fx(x^2+y^2)}{r^4}\epsilon^{-\iota\kappa r}\end{aligned}\right\} \quad\text{..............(50).}$$

It therefore follows, that the magnitude of the displacement represented by (50) is

$$\frac{F}{r}\epsilon^{-\iota\kappa r}\sin\phi\cos\theta \quad\text{.....................(51)},$$

and that its direction is along PT. Restoring the time factor, adding (48) and (51), and writing $cdS/2\lambda$ for F, we obtain

$$\frac{cdS}{2\lambda r}(1+\cos\theta)\sin\phi\cos\frac{2\pi}{\lambda}(bt-r),$$

which is Stokes' result.

We therefore see that Stokes' expression for the disturbance produced by an element of a plane wave of light is equivalent to the combination of a simple and a double source.

At the same time, if we were to carry out the investigation on the same lines as I have done in the case of sound in the paper referred to, there can, I think, be little doubt, that we should find that there is an infinite number of combinations of multiple sources which would produce the required effect, and consequently Stokes' law although the simplest, is only one out of an infinite number. The question is not, however, of very much importance in the case of light, inasmuch as, in problems relating to diffraction, we may with sufficient accuracy take $\sin\phi=\cos\theta=1$, in which case the disturbance due to the element will be

$$\frac{cdS}{\lambda r}\cos\frac{2\pi}{\lambda}(bt-r),$$

corresponding to the wave

$$w=c\sin\frac{2\pi}{\lambda}(bt-x).$$

Scattering of Light by Small Particles.

235. The physical explanation of the intensely blue colour of the sky, which cannot fail to have attracted the attention of those who have resided in warm countries, has formed the subject of various speculations. It has also been found by experiment, that a beam of light which is emitted by a bright cloud, exhibits decided traces of polarization, and that the direction of maximum polarization is perpendicular to that of the beam. The experi-

ments of Tyndall[1] on precipitated clouds, point to the conclusion, that both these phenomena are due to the existence of small particles of solid matter suspended in the atmosphere, which modify the waves of light in their course; and we shall now proceed to give an account of a theory due to Lord Rayleigh[2], by means of which these phenomena may be explained.

236. The theory of Lord Rayleigh in its original form, was an elastic solid theory; but it is equally applicable to the electromagnetic theory of light[3], since we shall hereafter see, that the equations, which are satisfied by the *electric displacement*, are of the same form as those which are satisfied by those portions of the displacements of an elastic solid, upon which distortion unaccompanied by dilatation depends.

If we suppose that the particles are spherical, it follows that when a plane wave of light impinges upon a particle, the latter will be thrown into a state of vibration; and the only possible motion which the particle can have, will consist of a motion of translation in the plane containing the directions of propagation and vibration of the impinging wave, and a motion of rotation about an axis perpendicular to this plane. If the ether be regarded as a medium, which possesses the properties of an elastic solid, the motion of the particles will give rise to two scattered waves, one of which will be a longitudinal wave, and therefore produces no optical effects, whilst the other will be a distortional wave, which will give rise to the sensation of light. If on the other hand, the ether be regarded as an electromagnetic medium, only one wave, viz. an optical wave, will be propagated. In order to obtain a complete mathematical solution, it would be necessary to introduce the boundary conditions[4], and to proceed on

[1] *Phil. Mag.* May 1869, p. 384.

[2] *Ibid.* Feb., April and June 1871; Aug. 1881.

[3] See *Phil. Mag.* Aug. 1881.

[4] If the ether be regarded as a medium which possesses the properties of an elastic solid, three suppositions may be made respecting the boundary conditions.

(i) We may suppose that no slipping takes place, which requires that the velocity of the ether in contact with the sphere should be equal to that of the sphere itself; but, inasmuch as there are reasons for thinking that the amplitudes of the vibrations of the matter are very much smaller than those of the ether in contact with it, except in the extreme case in which one of the free periods of the matter is equal to the period of the ethereal wave, this hypothesis is improbable.

(ii) We may suppose that partial slipping takes place. This hypothesis is

the same principles as in the corresponding acoustical problem[1]. It will not however be necessary to enter into any considerations of this kind, if we assume that the principal effect of the incident wave is to cause the particle to perform vibrations parallel to the direction of vibration of this wave.

237. To fix our ideas, let us suppose that the direction of propagation of the primary wave is vertical, and that the plane of vibration is the meridian. The particle will accordingly vibrate north and south, and its effect will be the same as that of a simple source of light, whose axis is in this direction. Accordingly if ϕ be the angle which any scattered ray makes with the line running north and south, it follows from (48), that the displacement will be of the form

$$\frac{F}{r}\sin\phi\cos\frac{2\pi}{\lambda}\left(bt-r\right),$$

and is therefore a maximum for rays, which lie in the vertical plane running east and west, for which $\phi=\frac{1}{2}\pi$; whilst there is no scattered ray along the north and south line for which $\phi=0$. If the primary wave is unpolarized, the light scattered north and south is entirely due to that component which vibrates east and west. Similarly any other ray scattered horizontally is perfectly polarized, and the vibration is performed in a horizontal plane. In other directions, the polarization becomes less and less complete as we approach the vertical, and in the vertical direction altogether disappears.

238. The preceding argument also shows, that the vibrations of polarized light must be perpendicular to the plane of polarization. For if the light scattered in a direction perpendicular to that of a primary wave be viewed through a Nicol's prism, it will be found that no light is transmitted, when the principal section is

open to the objection that the law of slipping is unknown, and would therefore involve an additional assumption; and also that it would introduce frictional resistance.

(iii) We may suppose that perfect slipping takes place. In this case the boundary conditions are continuity of normal motion, and zero tangential stress. This hypothesis has much to commend it on the ground of simplicity, since the action of the ether on the matter consists of a hydrostatic pressure, and in the case of a sphere is consequently reducible to a force; whereas, if no slipping or partial slipping took place, the action would (except in special cases) consist of a couple as well as a force.

[1] Lord Rayleigh, *Theory of Sound*, vol. II. § 334.

parallel to the direction of the primary wave. Hence the vibrations of the *extraordinary* wave in a uniaxal crystal, ***lie in the principal plane.***

239. We must now consider the colour of the scattered light. The experiments of Tyndall showed, that when the particles of foreign matter were sufficiently fine, the colour of the scattered light is blue. The simplest way of obtaining a theoretical explanation of this phenomenon, is by means of the method of dimensions. The ratio I of the amplitudes of the scattered and the primary light, is a simple number, and is therefore of no dimensions. This ratio must however be a function of T the volume of the disturbing particle, ρ' its density, r the distance of the point under consideration from it, b the velocity of propagation of light, and ρ the density of the ether. Since I is of no dimensions in mass, it follows that ρ and ρ' can only occur under the form ρ/ρ', which is a number and may be omitted; we have therefore to find out how I varies with T, r, λ and b.

Of these quantities b is the only one depending on the time; and therefore since I is of no dimensions in time, b cannot occur. We are therefore left with T, r and λ.

Now it is quite clear from dynamical considerations, that I varies directly as T and inversely as r, and must therefore be proportional to $T/\lambda^2 r$, T being of three dimensions in space. In passing from one part of the spectrum to another, λ is the only quantity which varies, and we thus obtain the important law:—

When light is scattered by particles, whose dimensions are small compared with the wave-length of light, the ratio of the amplitudes of the vibrations of the scattered and incident light, varies inversely as the square of the wave-length, and the ratio of the intensities, as the inverse fourth power.

From this law we see, that the intensity of the blue light is the greatest. Hence the blue colour of the sky may be accounted for on the supposition, that it is due to the action of minute particles of vapour, and also probably to the molecules of air, which scatter the waves proceeding from the sun.

Common Light.

240. The distinguishing feature of common light is, that it exhibits no trace of polarization; and the theory of sources of light given in § 232 furnishes an explanation of the reason why it is, that the light emitted from an incandescent substance is unpolarized.

The molecules of an incandescent body are in a violent state of vibration; each molecule may therefore be regarded as a centre of disturbance, which produces ethereal waves. The most general form of the waves produced by any molecule is given by (45), but for simplicity, we shall confine our attention to the first term of this series for which $n=0$. It therefore follows from (46), that at a distance from the molecule, which is large compared with the wave-length of light, the displacements would be represented by the equation

$$u = \left[\frac{A}{r} + \frac{1}{2r^3}\left\{Ar^2 - 3\,(Ax + By + Cz)\,x\right\}\right] \cos\frac{2\pi}{\lambda}(bt - r)$$

with symmetrical expressions for v and w, where A, B, C are proportional to the direction cosines of the direction of vibration of the molecule.

This expression represents a spherical wave of light, whose direction of vibration lies in the plane passing through the line of vibration of the molecule, and the line joining the latter with the eye of the observer.

But owing to a variety of causes, amongst which may be mentioned collisions, which are continually taking place between the molecules, the line of vibration of any particular molecule is perpetually changing, so that the angular motion of this line is most irregular. These changes take place in all probability with a rapidity, which is comparable with the period of waves of light, so that it is impossible for the eye to take cognizance of any particular direction. Moreover the light which is received from an incandescent body, is due to the superposition of the waves produced by an enormous number of vibrating molecules, the lines of vibration of each of which are different, and are continually changing. Hence the actual path which any particle of ether describes during a complete period is an irregular curve, whose form changes many million times in a second. We thus see why it is that common light is unpolarized.

241. We can now understand why interference fringes cannot be produced by means of light coming from two *different* sources. For the production of these fringes requires, that there should be a fixed relation between the phases of the two streams; but inasmuch as the two streams are affected by two distinct sets of irregularities, no fixed phase relation between them is possible. If however the two streams come from the same source, the irregularities by which the two streams are affected are identical, and consequently a fixed phase relation will exist between them.

EXAMPLES.

1. A luminous point is surrounded by an atmosphere containing a number of small equal particles of dust, the density of whose distribution varies inversely as the nth power of the distance from the point, and which scatter the light incident upon them. Show that except in the immediate vicinity of the luminous point, the $(n+1)/(n+3)$th part of the whole light scattered by the dust will be polarized.

2. Establish the truth of Stokes' expression for the effect of an element of an infinite plane wave at a point Q, by integration over the whole wave-front.

If the wave be finite, and all points of its boundary be at the same distance a from Q, prove that the displacement at Q will be

$$\left\{\sin\frac{2\pi}{\lambda}(vt-x)-\frac{1}{4a^3}(a+x)(a^2+x^2)\sin\frac{2\pi}{\lambda}(vt-a)\right\},$$

where x is the distance of Q from the wave at the plane of resolution.

3. In a biaxal crystal the ratios of the axes of the ellipsoid of elasticity are slightly different for different colours, so that the angles between the optic axes for yellow and violet are $\alpha, \alpha+\phi$. The normal to a wave-front of white light in such a crystal makes angles θ_1, θ_2 with the mean optic axis, and the planes through the normal and the optic axes make an angle ω with one another. Show that the directions of polarization lie within a small angle

$$\tfrac{1}{4}\phi\left(\frac{\sin\theta_1}{\sin\theta_2}-\frac{\sin\theta_2}{\sin\theta_1}\right)\frac{\sin\omega}{\sin\alpha}$$

CHAPTER XIV.

GREEN'S THEORY OF DOUBLE REFRACTION.

242. THE theory of double refraction proposed by Green[1], is the theory of the propagation of waves in an æolotropic elastic medium.

We have stated in Chapter XI., that the potential energy of such a medium is a homogeneous quadratic function of the six components of strain; and we shall now proceed to examine this statement.

Let O be any point of the medium, and let OA, OB, OC be the sides of an elementary parallelopiped of the medium when unstrained. Then any strain which acts upon the medium, will produce the following effects upon the element.

(i) Every point of the element will experience a bodily displacement.

(ii) The three sides OA, OB, OC will be elongated or contracted.

(iii) The element will be distorted into an oblique parallelopiped.

Let u, v, w be the component displacements at O; e, f, g the extensions of OA, OB, OC; a, b, c the angles which the faces OCA, OAB, OBC make with their original positions. Since a bodily displacement of the medium as a whole, cannot produce any strain, it follows that the potential energy due to strain cannot be a function of u, v, w; but since any displacement, which

[1] *Trans. Camb. Phil. Soc.* 1839; *Math. Papers*, p. 201.

produces an alteration of the forms (ii) or (iii) must necessarily endow the medium with potential energy, it follows that the potential energy due to strain, must be a function of the six strains e, f, g, a, b, c.

243. The most general form of the potential energy W, is given by the equation

$$W = W_1 + W_2 + W_3 + \ldots,$$

where W_n is a homogeneous n-tic function of the strains. It is evident that W cannot contain a constant term of the form W_0, for when the medium is unstrained, the potential energy is zero.

The most general expression for W_1 is

$$W_1 = Ee + Ff + Gg + Aa + Bb + Cc,$$

where $E, F \ldots$ are constants. Now Green supposed, that if the medium were subjected to external pressure, the first three terms of W_1 might come in; but it appears to me that this hypothesis is untenable. For if P be the stress of type e, then

$$P = \frac{dW}{de} = E + \frac{dW_2}{de} + \ldots,$$

accordingly if W contained a term W_1, stresses would exist, when the medium is free from strain. If the medium were *absolutely incompressible*, the stresses might undoubtedly contain terms independent of the strains. For if a portion of such a medium were enclosed in a rectangular box, and stresses E, F, G, A, B, C were applied to the sides of the box, of such magnitude as to preserve its rectangular form, no displacement, and consequently no strain would be produced, on account of the incompressibility of the medium; but the internal stresses would contain terms depending on the values of the surface stresses. These surface stresses could not however give rise to any terms in the potential energy, inasmuch as they do no work. If on the other hand, the medium were *compressible*, the effect of the surface stresses would be to produce displacements, and consequently strains depending upon them, in the interior of the medium; hence the internal stresses $P, Q, \ldots$ could not contain any terms independent of the strains, and the term W_1 could not exist. We have already pointed out, that in order to get rid of the pressural or dilatational wave, it is unnecessary to make the extravagant assumption, that the medium is incompressible; all that it is necessary to assume is, that the constants upon which compressibility depends, are very

large in comparison with those upon which distortion depends. Under these circumstances, we conclude that W_1 is zero, and that the internal stresses do not contain any terms independent of the strains. Also since the terms W_3, W_4... would introduce quadratic and cubic terms into the equations of motion, they will be neglected.

244. The potential energy is therefore a homogeneous quadratic function of the six strains, and accordingly contains twenty-one terms. Biaxal crystals, however, have three rectangular planes of symmetry; and as Green's object was to construct a theory which would explain double refraction, he assumed that the medium possessed this property. Whence the expression for W reduces to the following nine terms, and may be written

$$2W = Ee^2 + Ff^2 + Gg^2 + 2E'fg + 2F'ge + 2G'ef + Aa^2 + Bb^2 + Cc^2 \text{..............(1)},$$

where e, f, g, a, b, c are the six strains.

The coefficients in the expression for W are all constants, depending on the physical properties of the medium. The first three, E, F, G are called by Rankine[1] *coefficients of longitudinal elasticity*; the second three, E', F', G' are called *coefficients of lateral elasticity*; whilst the last three, A, B, C are the three *principal rigidities.*

245. The waves which are capable of being propagated in an isotropic medium, have already been shown to consist of two distinct types, which are propagated with different velocities; viz. longitudinal waves, which involve dilatation unaccompanied by distortion; and transversal waves, which involve distortion unaccompanied by dilatation. Waves of the first type depend upon the dilatation δ, and do not involve rotation; hence the rotations ξ, η, ζ are zero, and the displacements are the differential coefficients of a single function ϕ. Waves of the second type depend upon the rotations ξ, η, ζ, and do not involve dilatation; hence δ is zero, and the displacements must therefore satisfy the equation

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0,$$

which is the condition, that the displacement should be perpendicular to the direction of propagation.

[1] *Miscellaneous Scientific Papers*, p. 107.

246. Let us now consider a portion of a crystalline medium, which is bounded by a plane; and let plane waves whose vibrations are transversal, be incident normally upon the medium. The incident wave will produce a train of waves within the medium, which, as will presently be shown, will involve dilatation and distortion, unless certain relations exist between the coefficients. But since the disturbance which constitutes light, consists of a vector quantity, whose direction is perpendicular to the direction of propagation of the wave, it follows that the medium must be one, which is capable of propagating waves of transversal vibrations unaccompanied by waves of longitudinal vibrations. Green therefore assumed that the medium possessed this property, and investigated the relations which must exist between the coefficients, in order that this might be possible.

247. The equations of motion of the medium are

$$\rho\frac{d^2u}{dt^2}=\frac{dP}{dx}+\frac{dU}{dy}+\frac{dT}{dz}\quad\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

with two similar equations, where $P=dW/de$ &c. Substituting the values of P, Q ..., the equations of motion become

$$\left.\begin{aligned}
\rho\frac{d^2u}{dt^2}&=E\frac{d^2u}{dx^2}+C\frac{d^2u}{dy^2}+B\frac{d^2u}{dz^2}+(G'+C)\frac{d^2v}{dxdy}+(F'+B)\frac{d^2w}{dxdz}\\
\rho\frac{d^2v}{dt^2}&=C\frac{d^2v}{dx^2}+F\frac{d^2v}{dy^2}+A\frac{d^2v}{dz^2}+(E'+A)\frac{d^2w}{dydz}+(G'+C)\frac{d^2u}{dydx}\\
\rho\frac{d^2w}{dt^2}&=B\frac{d^2w}{dx^2}+A\frac{d^2w}{dy^2}+G\frac{d^2w}{dz^2}+(F'+B)\frac{d^2u}{dxdz}+(E'+A)\frac{d^2v}{dydz}
\end{aligned}\right\}(3).$$

Differentiate with respect to x, y, z and add, and we obtain

$$\rho\frac{d^2\delta}{dt^2}=E\frac{d^3u}{dx^3}+F\frac{d^3v}{dy^3}+G\frac{d^3w}{dz^3}+(2A+E')\frac{d^2a}{dydz}+(2B+F')\frac{d^2b}{dzdx}$$
$$+(2C+G')\frac{d^2c}{dxdy}.$$

If in this equation we put

$$\left.\begin{aligned}
&E=F=G=\mu\\
&E'=\mu-2A,\quad F'=\mu-2B,\quad G'=\mu-2C
\end{aligned}\right\}\ldots\ldots\ldots(4),$$

it becomes

$$\rho\frac{d^2\delta}{dt^2}=\mu\nabla^2\delta\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5).$$

Hence the relations between the coefficients which are given by (4), are the conditions that a longitudinal wave may be capable of being propagated through the medium, unaccompanied by

transversal waves; and therefore if these conditions are satisfied, longitudinal waves will be propagated through the medium with a velocity $(\mu/\rho)^{\frac{1}{2}}$.

By means of (4), the equations of motion may now be written

$$\left.\begin{aligned}\rho\frac{d^2u}{dt^2} &= \mu\frac{d\delta}{dx} + B\frac{d\eta}{dz} - C\frac{d\zeta}{dy}\\ \rho\frac{d^2v}{dt^2} &= \mu\frac{d\delta}{dy} + C\frac{d\zeta}{dx} - A\frac{d\xi}{dz}\\ \rho\frac{d^2w}{dt^2} &= \mu\frac{d\delta}{dz} + A\frac{d\xi}{dy} - B\frac{d\eta}{dx}\end{aligned}\right\} \quad (6),$$

from which we deduce,

$$\left.\begin{aligned}\rho\frac{d^2\xi}{dt^2} &= A\nabla^2\xi - \frac{d\Omega}{dx}\\ \rho\frac{d^2\eta}{dt^2} &= B\nabla^2\eta - \frac{d\Omega}{dy}\\ \rho\frac{d^2\zeta}{dt^2} &= C\nabla^2\zeta - \frac{d\Omega}{dz}\end{aligned}\right\} \quad (7),$$

where

$$\Omega = A\frac{d\xi}{dx} + B\frac{d\eta}{dy} + C\frac{d\zeta}{dz} \quad \ldots\ldots\ldots\ldots\ldots (8),$$

and the expression for the potential energy becomes

$$2W = \mu(e+f+g)^2 + A(a^2 - 4fg) + B(b^2 - 4ge) + C(c^2 - 4ef) \ldots (9).$$

The stresses are given by the equations

$$\left.\begin{aligned}P &= \mu\delta - 2(Cf + Bg)\\ Q &= \mu\delta - 2(Ag + Ce)\\ R &= \mu\delta - 2(Be + Af)\\ S &= Aa, \quad T = Bb, \quad U = Cc\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots (10).$$

248. Equations (6) and (7) show, that the special kind of æolotropic medium considered by Green, is capable of propagating two distinct types of waves, viz. dilatational waves, whose velocity of propagation has been shown to be equal to $(\mu/\rho)^{\frac{1}{2}}$, and distortional waves, whose velocity of propagation is determined by (7). We shall presently show, that the velocity of propagation of the distortional waves, is determined by the same quadratic equation as in Fresnel's theory; but previously to doing this, it will be desirable to consider a little more closely the properties of the medium.

249. In a crystalline medium, which possesses three rectangular planes of symmetry, the shearing stress across any plane which is not a principal plane, will in general be a function of the extensions as well as of the shearing strain parallel to that plane. It is however possible for a medium to be symmetrical, as regards rigidity, with respect to each of the three principal axes:—in other words, the medium may be such, that if any plane be drawn parallel to one of the principal axes (say x), and S_1, a_1 be the shearing stress and strain parallel to that plane and perpendicular to the axis of x, then $S_1 = Aa_1$. We shall now show, that when a medium possesses this property, the relations (4) must exist between the coefficients.

Let Ox, Oy, Oz be the axes of crystalline symmetry; and let BC be the intersection of any plane parallel to Ox with the plane yz; and consider a portion of the medium, which is bounded by the plane BC and two fixed rigid planes perpendicular to Ox. Draw Oy_1, Oz_1 respectively perpendicular and parallel to BC, and let the suffixed letters denote the values of corresponding quantities referred to Ox, Oy_1, Oz_1 as axes.

If θ be the angle which Oy_1 makes with Oy, then

$$S_1 = S\cos 2\theta + \tfrac{1}{2}(R - Q)\sin 2\theta.$$

Also, since the medium is supposed to be bounded by two rigid planes perpendicular to Ox, there can be no extension nor contraction parallel to Ox, whence

$$Q = Ff + E''g, \quad R = E'f + Gg;$$

accordingly,

$$S_1 = Aa\cos 2\theta + \tfrac{1}{2}\{(E' - F)f + (G - E'')g\}\sin 2\theta \ldots (11).$$

But, if $m = \cos\theta$, $n = \sin\theta$,

$$f = \left(m\frac{d}{dy_1} - n\frac{d}{dz_1}\right)(mv_1 - nw_1)$$

$$= m^2 f_1 + n^2 g_1 - mna_1;$$

also
$$g = n^2 f_1 + m^2 g_1 + mna_1.$$

Again,

$$a = \frac{dw}{dy} + \frac{dv}{dz}$$

$$= \left(m\frac{d}{dy_1} - n\frac{d}{dz_1}\right)(nv_1 + mw_1) + \left(n\frac{d}{dy_1} + m\frac{d}{dz_1}\right)(mv_1 - nw_1)$$

$$= a_1\cos 2\theta + (f_1 - g_1)\sin 2\theta.$$

Substituting in (11), we obtain

$$S_1 = \{a_1 A \cos^2 2\theta + \tfrac{1}{4}(G+F-2E')\sin^2 2\theta\}$$
$$+\tfrac{1}{2}(f_1-g_1)\{A - \tfrac{1}{4}(G+F-2E)\}\sin 4\theta$$
$$+\tfrac{1}{4}(f_1+g_1)(G-F)\sin 2\theta.$$

It therefore follows that if

$$G = F = \mu,$$
$$A = \tfrac{1}{4}(G+F-2E') = \tfrac{1}{2}(\mu - E'),$$

we shall have $$S_1 = Aa_1.$$

In a similar way it can be shown, that in order that $T_1 = Bb_1$, and $U_1 = Cc_1$, we must have

$$E = F = G = \mu,$$
$$2A = \mu - E', \quad 2B = \mu - F', \quad 2C = \mu - G',$$

which are equivalent to (4).

If therefore a portion of the medium considered by Green, which is bounded by two fixed planes perpendicular to any one of the principal axes, be subjected to a shearing stress whose direction is perpendicular to that axis, and which lies in *any* plane parallel to that axis, the ratio of the shearing stress to the shearing strain is equal to the principal rigidity corresponding to that axis. Moreover a crystalline medium which possesses this property, also possesses the property of being able to transmit waves of transversal vibrations unaccompanied by waves of longitudinal vibrations. Hence the relations which Green supposed to exist between the nine constants, are not mere adventitious relations, which were assumed for the purpose of obtaining a particular analytical result, but correspond to and specify a particular physical property of the medium.

250. We shall now show, that the velocity of propagation of the distortional waves is determined by Fresnel's law.

To satisfy (7) let u, v, w be those portions of the displacements upon which distortion depends; let l, m, n be the direction cosines of the wave-front, and λ, μ, ν those of the direction of vibration. Then we may assume that

$$u = S\lambda, \quad v = S\mu, \quad w = S\nu,$$

when $$S = \epsilon^{\iota\kappa(lx+my+nz-Vt)}.$$

From these equations combined with (7) of § 187, we obtain

$$\xi = \iota\kappa(m\nu - n\mu)S, \quad \eta = \iota\kappa(n\lambda - l\nu)S, \quad \zeta = \iota\kappa(l\mu - m\lambda)S.$$

If we put

$$\lambda' = m\nu - n\mu, \quad \mu' = n\lambda - l\nu, \quad \nu' = l\mu - m\lambda \text{......(12)},$$

so that λ', μ', ν' are the direction cosines of the rotation, and then substitute the values of ξ, η, ζ in (7), we obtain

$$\left.\begin{aligned}(\rho V^2 - A)\lambda' + (Al\lambda' + Bm\mu' + Cn\nu')\,l &= 0\\ (\rho V^2 - B)\mu' + (Al\lambda' + Bm\mu' + Cn\nu')\,m &= 0\\ (\rho V^2 - C)\nu' + (Al\lambda' + Bm\mu' + Cn\nu')\,n &= 0\end{aligned}\right\} \text{.(13)},$$

whence
$$\frac{l^2}{\rho V^2 - A} + \frac{m^2}{\rho V^2 - B} + \frac{n^2}{\rho V^2 - C} = 0 \text{...........(14)}.$$

From (12) we at once deduce

$$\left.\begin{aligned}l\lambda + m\mu + n\nu &= 0\\ l\lambda' + m\mu' + n\nu' &= 0\end{aligned}\right\} \text{....................(15)}.$$

It follows from (14), that the velocities of propagation of the two waves within the crystal are determined by the same quadratic as in Fresnel's theory, and that the wave surface is Fresnel's.

251. From (12) it follows, that the direction of displacement and rotation are in the front of the wave, and also that these directions are at right angles to one another.

Multiplying (13) by λ', μ', ν' and adding, we obtain

$$\rho V^2 = A\lambda'^2 + B\mu'^2 + C\nu'^2 \text{.................(16)},$$

which shows that the velocity of propagation of either wave, is inversely proportional to the length of that radius vector of the ellipsoid

$$Ax^2 + By^2 + Cz^2 = 1,$$

which is parallel to the direction of rotation.

We also obtain from (13)

$$(\rho V^2 - A)\lambda'/l = (\rho V^2 - B)\mu'/m = (\rho V^2 - C)\nu'/n \text{......(17)}.$$

252. Writing $a^2 = A/\rho$, $b^2 = B/\rho$, $c^2 = C/\rho$, we see from (19) of §109, that if x, y, z be the coordinates of the point of contact of the tangent plane to the wave surface, which is parallel to the wave-front, then

$$x = lV\,(r^2 - A/\rho)/(V^2 - A/\rho),$$
$$y = mV\,(r^2 - B/\rho)/(V^2 - B/\rho),$$
$$z = nV\,(r^2 - C/\rho)/(V^2 - C/\rho),$$

and therefore by (17)

$$(r^2 - A/\rho)\lambda'/x = (r^2 - B/\rho)\mu'/y = (r^2 - C/\rho)\nu'/z \text{......(18)},$$

from which it follows, as in § 112, that the direction of rotation in any wave, coincides with that of the projection of the ray on the tangent plane to the wave surface, which is parallel to the wave. Now the direction of displacement is perpendicular to that of rotation, and therefore Green's theory requires us to suppose, that the vibrations of polarized light are parallel to the plane of polarization.

From (11) we deduce

$$l^{-1}(\rho V^2 - A)(m\nu - n\mu) = m^{-1}(\rho V^2 - B)(n\lambda - l\nu)$$
$$= n^{-1}(\rho V^2 - C)(l\mu - m\lambda),$$

and since $$l\lambda + m\mu + n\nu = 0$$

we obtain $$\frac{l}{\lambda}(B - C) + \frac{m}{\mu}(C - A) + \frac{n}{\nu}(A - B) = 0 \quad\text{.........(19)},$$

which determines the direction of vibration.

253. The theory of Green, although dynamically sound, renders it necessary to suppose that the vibrations of polarized light are parallel to the plane of polarization, which is one objection; also if we disregard this difficulty, another difficulty crops up in applying the theory to crystalline reflection and refraction, owing to the necessity of making some assumption, involving relations between the physical constants of isotropic and crystalline media.

To investigate this point, let us consider the reflection and refraction of light at the surface of a uniaxal crystal, whose face is perpendicular to the axis. In order that the incident light should give rise to an extraordinary wave, it is necessary on this theory, to suppose that the incident vibrations are perpendicular to the plane of incidence.

In the first medium, the equation of motion is

$$\rho \frac{d^2w}{dt^2} = n\left(\frac{d^2w}{dy^2} + \frac{d^2w}{dx^2}\right) \quad\text{....................(20)},$$

and in the crystal,

$$\rho_1 \frac{d^2w_1}{dt^2} = a^2 \frac{d^2w_1}{dy^2} + c^2 \frac{d^2w_1}{dx^2} \quad\text{....................(21)},$$

where we have written a^2, c^2 for A and C.

Let $$w = A\epsilon^{\iota\kappa(-x\cos i + y\sin i - Vt)} + A'\epsilon^{\iota\kappa(x\cos i + y\sin i - Vt)},$$
$$w_1 = A_1\epsilon^{\iota\kappa_1(-x\cos r + y\sin r - V_1 t)},$$

where $$\kappa \sin i = \kappa_1 \sin r, \quad \kappa V = \kappa_1 V_1 \quad\text{..............(22)}.$$

From (20) we obtain $\quad V^2 = n/\rho,$

and from (21) $\quad V_1^2 = (a^2 \sin^2 r + c^2 \cos^2 r)/\rho_1.$

The surface conditions for continuity of displacement and stress give

$$w = w_1, \quad n\frac{dw}{dx} = c^2\frac{dw_1}{dx},$$

when $x = 0$; whence $\quad A + A' = A_1,$

$$\kappa n (A - A') \cos i = A_1 c^2 \kappa_1 \cos r,$$

the last of which, by (22), becomes

$$A - A' = A_1 \frac{c^2 \tan i}{n \tan r},$$

whence
$$A' = A \frac{n \tan r - c^2 \tan i}{n \tan r + c^2 \tan i} \qquad (23),$$

$$A_1 = \frac{2An \tan r}{n \tan r + c^2 \tan i} \qquad (24).$$

We have hitherto avoided assuming, that any relations exist between the physical constants of the two media; but, in order that these results should be consistent with those which the theory furnishes for isotropic media, it would be necessary to suppose that $n = c^2$; and the formulæ then show that the amplitudes of the reflected and refracted light would be the same as if the crystal were an isotropic medium. Since the wave whose velocity is c is refracted according to the ordinary law, the assumption that $n = c^2$ might at first sight appear to be a plausible one in the case of uniaxal crystal; but, if we attempt to apply the theory to biaxal crystals, there is no valid reason why n should be assumed to be equal to one of the three principal rigidities, rather than to either of the other two.

If we adopt the assumption of MacCullagh and Neumann, that $\rho = \rho_1$, the intensities will be proportional to the square roots of the amplitudes, and we shall obtain

$$A' = A \frac{(a^2 \sin^2 r + c^2 \cos^2 r) \sin 2i - c^2 \sin 2r}{(a^2 \sin^2 r + c^2 \cos^2 r) \sin 2i + c^2 \sin 2r},$$

$$A_1 = \frac{2A\,(a^2 \sin^2 r + c^2 \cos^2 r) \sin 2i}{(a^2 \sin^2 r + c^2 \cos^2 r) \sin 2i + c^2 \sin 2r}.$$

The formulæ, as will be shown hereafter, agree with the expressions found for the intensity on the electromagnetic theory;

but Lord Rayleigh has shown, that the assumption that the densities are equal is not a legitimate one in the case of two isotropic media, since it leads to two polarizing angles, and there can be little doubt that, in the case of crystalline media, the same assumption would lead to a similar result, and would therefore be one which it is not permissible to make. It thus appears that Green's theory fails to furnish a satisfactory explanation of crystalline reflection and refraction.

To work out a rigorous theory of the reflection and refraction of waves, at the surface of separation of an isotropic medium, and an æolotropic medium such as Green's, on the supposition that the velocities of propagation of the dilatational or pressural waves in both media, are very great in comparison with the velocities of propagation of the distortional waves, would be a mere question of mathematics, and could be effected without difficulty on the lines of Green's and Lord Rayleigh's investigations, when both media are isotropic. But the only physical interest of such investigations lies in their ability (or inability) to explain optical phenomena; and therefore, having regard to the failure of Green's theory to furnish satisfactory results in the case of crystalline reflection and refraction, it seems scarcely worth while to pursue such investigations.

254. The theory of Green stands on a perfectly sound dynamical basis, and the various suppositions which he has made with regard to the relations between the constants, are not adventitious assumptions made for the purpose of deducing Fresnel's wave surface, but correspond to definite physical properties of the medium. The assumption, that the medium possesses three rectangular planes of symmetry, is necessary, in order to account for the fact, that in biaxal crystals, there are three perpendicular directions, in which a ray of light can be transmitted without division. Also since the phenomenon of polarization can only be explained on the supposition, that the disturbance which produces optical effects is a vector, whose direction is perpendicular to that of the propagation of the wave, it is necessary to suppose, that the medium is one which is capable of transmitting distortional vibrations independently of dilatational vibrations; and the conditions for this require, that certain relations should exist between the constants, which are given by equations (4), and

which reduce the expression for the potential energy to four terms. It is no doubt the case, that when waves of light whose vibrations lie in the plane of incidence, are reflected and refracted by a crystal, waves of longitudinal vibrations would be excited; but this difficulty might be evaded, by supposing that μ is very large compared with A, B and C. The theory accordingly at first sight appears to be a very promising one; but, as we have already shown, there are strong grounds for believing, that the vibrations of polarized light are perpendicular instead of parallel to the plane of polarization; and the circumstance, that Green's theory requires us to adopt the latter hypothesis, is one of the principal reasons which has prevented it from being accepted as the true theory.

255. Attention has been called to the fact, that the potential energy contains cubic and higher terms, which have been neglected. Glass, however, and most transparent isotropic media exhibit double refraction, when under the influence of stress; and this fact shows, that the propagation of ethereal waves is modified, when the medium is subjected to stress. A theory which would take into account the effect of these external stresses, and might also throw light on double refraction, could be constructed as follows.

The quantity e is the extension parallel to x, and to a first approximation its value is du/dx; if however the approximation were carried a stage further, it would be found that the strains contain quadratic terms. Accordingly if the more complete values of the strains were substituted in (1), they would give rise to cubic terms in W_2. Moreover in this case, it would be necessary to take W_3 into account; but in forming the expression for this quantity, it would be sufficient to take $e = du/dx$, $f = dv/dy$, &c. The final equations of motion would accordingly contain quadratic as well as linear terms. The solution of these equations would then have to be conducted on the same principles, as the well-known problem of the propagation of waves in a liquid, which has a motion independent of the wave motion. In the first place, let u_1, v_1, w_1 be the statical portions of the displacements, which depend upon the external stresses; and let these quantities be found from the complete equations of equilibrium. Next let u_2, v_2, w_2 be the portions of the displacements due to the wave motion, so that $u_1 + u_2$, $v_1 + v_2$, $w_1 + w_2$ are the total displacements of the medium

when in motion; and let these quantities be substituted in the equations of motion, neglecting quadratic terms of the form u_2^2 &c. We should thus obtain three linear equations for determining u_2, v_2, w_2, into which the external stresses would have been introduced. So far as I am aware, a theory of this kind has not been worked out, but it would be interesting to examine the results to which it leads in some simple case.

CHAPTER XV.

THEORY OF LORD RAYLEIGH AND SIR W. THOMSON.

256. THE theory which we shall now consider, was first suggested by Rankine, but was subsequently proposed and developed independently by Lord Rayleigh[1]. The theory might be regarded as one, which depends upon the mutual reaction of ether and matter; but inasmuch as it is capable of explaining several important phenomena, it will be desirable to consider it at once.

We have already pointed out the unsatisfactory character of Green's theory, when applied to double refraction. We have moreover seen, that there are strong grounds for supposing, that the rigidity of the ether is the same in all isotropic media, and that reflection and refraction are due to a difference of density. The properties of an isotropic medium are the same in all directions, but those of a crystal in any direction depend upon the inclination of that direction to the axes of symmetry of the crystal. Lord Rayleigh therefore assumed, that the two elastic constants of the ether are the same in crystalline as in isotropic media; but that owing to the peculiar structure of the matter composing the crystal, the ether behaves as if its density were æolotropic.

257. Since the density of every medium is a scalar function, it might appear that this assumption involves a physical impossibility; but it is easy to give an example of a system which behaves in this manner. Let an ellipsoid, suspended by a fine wire, perform small oscillations without rotation in an infinite liquid. If U, V, W be the velocities of the ellipsoid parallel to

[1] Hon. J. W. Strutt, *Phil. Mag.*, June, 1871.

its axes, its kinetic energy will be equal to $\frac{1}{2}M(U^2 + V^2 + W^2)$; the kinetic energy of the liquid is equal to $\frac{1}{2}(P'U^2 + Q'V^2 + R'W^2)$; and therefore the kinetic energy of the solid and liquid will be of the form $\frac{1}{2}(PU^2 + QV^2 + RW^2)$. Hence the effect of the liquid is to cause the ellipsoid to oscillate in the same manner as a particle, whose density is a function of its direction of motion. We thus have an example of a system, which behaves as if its density were æolotropic.

The point may also be considered from a somewhat different aspect. In the hydrodynamical problem, the resultant pressure exerted by the liquid, consists of three components $P'\dot{U}$, $Q'\dot{V}$, $R'\dot{W}$. Lord Rayleigh's hypothesis is therefore equivalent to the assumption, that the effect of matter upon ether, is represented by a force whose components are $-\rho_x\ddot{u}$, $-\rho_y\ddot{v}$, $-\rho_z\ddot{w}$ parallel to the axes of crystalline symmetry, where u, v, w denote the displacements of the ether; in other words, these forces are proportional to the component accelerations of the ether. In a biaxal crystal, ρ_x, ρ_y, ρ_z are all different; but in an isotropic medium they are equal.

258. The kinetic energy of the ether may accordingly be taken to be equal to

$$\tfrac{1}{2}\iiint(\rho_x\dot{u}^2 + \rho_y\dot{v}^2 + \rho_z\dot{w}^2)\,dx\,dy\,dz\,;$$

whilst the potential energy is the same as in an isotropic medium. And by employing the Principle of Least Action, or the Principle of Virtual Work, the equations of motion will be found to be

$$\left.\begin{aligned}\rho_x\frac{d^2u}{dt^2} &= (A - B)\frac{d\delta}{dx} + B\nabla^2u\\ \rho_y\frac{d^2v}{dt^2} &= (A - B)\frac{d\delta}{dy} + B\nabla^2v\\ \rho_z\frac{d^2w}{dt^2} &= (A - B)\frac{d\delta}{dz} + B\nabla^2w\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots(1),$$

where $A - B = m = k + \frac{1}{3}n$, $B = n$, m and n being the elastic constants in Thomson and Tait's notation.

259. Before entering into any further discussion respecting this theory, it will be desirable to solve these equations, in order to find out what they lead us to. We shall accordingly proceed to determine the velocity of propagation[1].

[1] Glazebrook, *Phil. Mag.* (5), Vol. XXVI. p. 521.

Let $\quad u = S\lambda, \quad v = S\mu, \quad w = S\nu, \quad S = \epsilon^{\iota\kappa(lx+my+nz-Vt)}$ ...(2).

Substituting in (1), we obtain

$$\left.\begin{aligned}\lambda\rho_x V^2 &= (A - B)(l\lambda + m\mu + n\nu)\,l + B\lambda\\ \mu\rho_y V^2 &= (A - B)(l\lambda + m\mu + n\nu)\,m + B\mu\\ \nu\rho_z V^2 &= (A - B)(l\lambda + m\mu + n\nu)\,n + B\nu\end{aligned}\right\}\text{.........(3).}$$

Let $\quad a^2 = B/\rho_x, \quad b^2 = B/\rho_y, \quad c^2 = B/\rho_z$ ....(4),

then if we multiply (3) in order, by l, m, n and add, we shall obtain

$$BV^2\left(\frac{l\lambda}{a^2} + \frac{m\mu}{b^2} + \frac{n\nu}{c^2}\right) = A\,(l\lambda + m\mu + n\nu)\text{.........(5).}$$

Transposing the terms $B\lambda$ &c. in (3) to the left-hand side, multiplying by l, m, n, dividing by $V^2 - a^2$ &c., and adding, we obtain

$$B\left(\frac{l\lambda}{a^2} + \frac{m\mu}{b^2} + \frac{n\nu}{c^2}\right)$$

$$= (A - B)\,(l\lambda + m\mu + n\nu)\left(\frac{l^2}{V^2 - a^2} + \frac{m^2}{V^2 - b^2} + \frac{n^2}{V^2 - c^2}\right)\text{...(6),}$$

and therefore by (5),

$$\frac{l^2}{V^2 - a^2} + \frac{m^2}{V^2 - b^2} + \frac{n^2}{V^2 - c^2} = \frac{A}{(A - B)\,V^2}\text{...........(7).}$$

This is a cubic equation for determining the velocity of propagation, and shows that corresponding to a given direction, the medium is capable of propagating three waves.

By means of (4), equations (3) may be written in the form

$$V^2 - a^2 = (A - B)(l\lambda + m\mu + n\nu)\,a^2 l/\lambda B,$$

with two similar equations; whence we readily obtain

$$\frac{a^2 l}{\lambda}(b^2 - c^2) + \frac{b^2 m}{\mu}(c^2 - a^2) + \frac{c^2 n}{\nu}(a^2 - b^2) = 0\text{........(8).}$$

260. Now $A = k + \frac{4}{3}n$, and $B = n$, where k and n are the resistance to compression and the rigidity respectively; and according to the views of Green, the resistance to compression of the ether is very much greater than its rigidity, and therefore A is exceedingly large compared with B. If therefore in (7) we put $B/A = 0$, and multiply the right-hand side of (7) by $l^2 + m^2 + n^2$, this equation may be put into the form

$$\frac{l^2 a^2}{V^2 - a^2} + \frac{m^2 b^2}{V^2 - b^2} + \frac{n^2 c^2}{V^2 - c^2} = 0\text{..............(9).}$$

This equation determines the velocity of propagation of the two optical waves. The velocity of propagation of the longitudinal wave is infinite.

The direction of vibration is determined by (8) together with the equation

$$l\lambda + m\mu + n\nu = 0 \text{.......................} (10).$$

Equation (9) does not lead to Fresnel's wave surface; and inasmuch as the experiments of Glazebrook[1] have shown, that Fresnel's wave surface is a very close approximation to the truth, the theory in its present form is unsatisfactory. We shall therefore proceed to consider a modification of this theory which has been proposed by Sir W. Thomson[2], by means of which Fresnel's wave surface can be obtained.

261. When a disturbance is communicated to a homogeneous isotropic elastic medium, two waves are propagated from the centre of disturbance with different velocities; one of which is a wave of dilatation, whose vibrations are perpendicular to the wave-front, and whose velocity of propagation is equal to $(k+\frac{4}{3}n)^{\frac{1}{2}}/\rho^{\frac{1}{2}}$; whilst the other is a distortional wave, which does not involve dilatation, and whose velocity of propagation is equal to $n^{\frac{1}{2}}/\rho^{\frac{1}{2}}$.

In applying the theory of elastic media to explain optical phenomena, it is necessary to get rid of the difficulty which arises from the fact, that such media are capable of propagating dilatational waves. This may be done by supposing that the ratio $(k+\frac{4}{3}n)^{\frac{1}{2}}/n^{\frac{1}{2}}$, of the velocity of propagation of the dilatational wave to that of the distortional wave, is either very large or very small; which requires either that k should be very large compared with n, or should be very nearly equal to $-\frac{4}{3}n$. Green adopted the former supposition, on the ground that, if the latter were true, the medium would be unstable. Sir W. Thomson, however, has pointed out that, if k is negative and numerically less than $\frac{4}{3}n$, the medium will be stable, *provided we either suppose the medium to extend all through boundless space, or give it a fixed containing vessel as a boundary.*

Putting $\qquad U=(k+\frac{4}{3}n)^{\frac{1}{2}}/\rho^{\frac{1}{2}}, \quad V=n^{\frac{1}{2}}/\rho^{\frac{1}{2}},$

it is obvious, that if a small disturbance be communicated to the medium, U will be real, provided $k+\frac{4}{3}n$ be positive, and therefore

[1] *Phil. Trans.* 1879, p. 287; 1880, p. 421.
[2] *Phil. Mag.* (5), Vol. XXVI. p. 414.

the motion will not increase indefinitely with the time, but will be periodic; but, if $k + \frac{4}{3}n$ be negative, U will be imaginary, in which case the disturbance will either increase or diminish indefinitely with the time, and the medium will either explode or collapse, and will therefore be thoroughly unstable. If $k = -\frac{4}{3}n$, U will be zero, and therefore the medium will be incapable of propagating a dilatational wave. The principal difficulty in adopting this hypothesis appears to me to arise from the fact, that it requires us to suppose that the compressibility is negative:—in other words, that an increase of pressure produces an increase of volume. So far as I am aware, no medium with which we are acquainted possesses this property; and it is very difficult to form a mental representation of such a medium. On the other hand, there does not appear to be any *a priori* reason for supposing, that a medium possessing this property does not exist; if, therefore, we adopt Sir W. Thomson's hypothesis, it follows that elastic media may be classed under the following three categories:—(i) media which contract under pressure, for which k may have any positive value; (ii) media which expand but do not explode or collapse under pressure, for which k may have any negative value which is numerically less than $\frac{4}{3}n$; (iii) media which explode or collapse under pressure, for which k may have any negative value which is numerically greater than $\frac{4}{3}n$.

262. In order to explain more clearly the necessity of supposing, that the medium extends through infinite space, or is contained with rigid boundary, we observe that the potential energy is equal to

$$\tfrac{1}{2}\iiint\{(m+n)\,\delta^2 + n\,(a^2+b^2+c^2) - 4n\,(ef+fg+ge)\}\,dxdydz.$$

Integrating the last term by parts, it becomes

$$2n\iiint\left(\frac{dv}{dz}\frac{dw}{dy} + \frac{dw}{dx}\frac{du}{dy} + \frac{du}{dy}\frac{dv}{dx}\right)dxdydz - \text{a surface integral.}$$

If the boundary is fixed, or at an infinite distance, u, v, w must be zero at the boundary, whence the surface integral vanishes; accordingly

$$W = \tfrac{1}{2}\iiint\{(m+n)\,\delta^2 + n\,(\xi^2+\eta^2+\zeta^2)\}\,dxdydz.$$

The value of W is positive when $m+n$ is positive, i.e. when $k > -\frac{4}{3}n$; in other words, work will have to be done in order to bring the medium into its strained condition.

263. We shall now develop the consequences of supposing that $m+n$ or A is so exceedingly small, that it may be treated as zero.

In the first place, the right-hand side of (7) is zero, and therefore the velocity of propagation is determined by Fresnel's equation, and accordingly the wave surface is Fresnel's.

Since B is not zero, equations (6) and (7) show that

$$\frac{l\lambda}{a^2}+\frac{m\mu}{b^2}+\frac{n\nu}{c^2}=0 \quad\text{.....................(11).}$$

Equations (3) may be written

$$\left.\begin{aligned}\lambda(V^2/a^2-1)&=-(l\lambda+m\mu+n\nu)\,l\\ \mu(V^2/b^2-1)&=-(l\lambda+m\mu+n\nu)\,m\\ \nu(V^2/c^2-1)&=-(l\lambda+m\mu+n\nu)\,n\end{aligned}\right\}\quad\text{............(12).}$$

Multiplying these equations by λ/a^2, μ/b^2, ν/c^2, adding, and taking account of (11), we obtain

$$V^2(\lambda^2/a^4+\mu^2/b^4+\nu^2/c^4)=\lambda^2/a^2+\mu^2/b^2+\nu^2/c^2 \quad\text{......(13),}$$

which gives V in terms of the direction of vibration.

Again from (12), we obtain

$$(V^2-a^2)\lambda/a^2l=(V^2-b^2)\mu/b^2m=(V^2-c^2)\nu/c^2n=H\ \text{(say)...(14),}$$

which determine the direction (λ, μ, ν) of the vibrations corresponding to a given wave-front.

Equation (7) with A very small, but not zero, shows that a quasi-dilatational wave will be propagated, whose velocity is very small; if therefore in (14), V denote the velocity of this wave, V will be very nearly zero, and consequently the direction of vibration will be approximately determined by the equations

$$\lambda/l=\mu/m=\nu/n \quad\text{.......................(15),}$$

which shows that the direction of vibration in this wave is sensibly perpendicular to the wave-front.

264. We have also shown, (19) of § 109, that in Fresnel's wave surface

$$x=\frac{lV(r^2-a^2)}{V^2-a^2}=\lambda V(r^2-a^2)/Ha^2 \quad\text{............(16),}$$

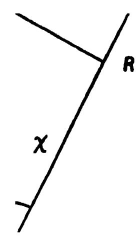
by (14), with two similar equations, accordingly

$$\lambda x+\mu y+\nu z=\frac{H}{V}\left(\frac{a^2x^2}{r^2-a^2}+\frac{b^2y^2}{r^2-b^2}+\frac{c^2z^2}{r^2-c^2}\right)=0\text{...(17),}$$

whence the direction of vibration is perpendicular to the ray.

265. Since the equation $l\lambda + m\mu + n\nu = 0$ is not satisfied, the direction of vibration does not lie in the wave-front. We shall now show, that it is determined by the following construction.

Let P be the point where a ray proceeding from a point O within the crystal, meets the wave surface whose centre is O; let PY be the tangent plane to the wave-surface at P, OY the



perpendicular on it from O, and draw YR perpendicular to OP. Then RY is the direction of vibration.

To prove this, let L, M, N be the direction cosines of RY, then

$$-RY \,.\, L = OR \,.\, x/r - OY \,.\, l.$$

But $OY = V$, and $OR = OY^2/OP = V^2/r$; whence

$$\begin{aligned} L \,.\, RY &= lV - V^2x/r^2 \\ &= lV\left\{1 - \frac{V^2(r^2 - a^2)}{r^2(V^2 - a^2)}\right\} \\ &= \frac{lVa^2(V^2 - r^2)}{r^2(V^2 - a^2)} \\ &= \frac{\lambda V(V^2 - r^2)}{Hr^2} \end{aligned}$$

by (14), whence

$$L/\lambda = M/\mu = N/\nu \text{..................} (18).$$

In Fresnel's theory, PY is the direction of vibration; but although on this theory the direction of vibration is not the same as in Fresnel's theory, yet it lies in the plane containing the ray and the wave normal, and therefore the vibrations of polarized light on emerging from the crystal, are perpendicular to the plane of polarization.

266. If L', M', N' are the direction cosines of PY, then

$$-L'.PY = x - lV$$
$$= \frac{lV(r^2 - V^2)}{V^2 - a^2}$$
$$= \lambda V(r^2 - V^2)/Ha^2$$

by (14); whence

$$\frac{L'}{\lambda/a^2} = \frac{M'}{\mu/b^2} = \frac{N'}{\nu/c^2} \quad \text{.....................(19).}$$

267. The following expression for $\cos\chi$ is also useful. We have

$$\cos\chi = \cos PYR = LL' + MM' + NN'$$
$$= \frac{\lambda^2/a^2 + \mu^2/b^2 + \nu^2/c^2}{(\lambda^2/a^4 + \mu^2/b^4 + \nu^2/c^4)^{\frac{1}{2}}}$$

by (18) and (19). But

$$\cos\chi = \frac{V}{r} = \frac{1}{r}\left(\frac{\lambda^2/a^2 + \mu^2/b^2 + \nu^2/c^2}{\lambda^2/a^4 + \mu^2/b^4 + \nu^2/c^4}\right)^{\frac{1}{2}},$$

whence

$$\frac{1}{r^2} = \frac{\lambda^2}{a^2} + \frac{\mu^2}{b^2} + \frac{\nu^2}{c^2} = \frac{\cos^2\chi}{V^2} \quad \text{..................(20).}$$

268. Another point of importance is, that according to this theory, it is necessary to suppose that the rigidity is the same in crystalline as in isotropic media; and therefore that refraction is due to a difference of density. For if we consider two different media bounded by the plane $x = 0$, the displacements u, v, w must be continuous. Now the continuity of v and w when $x = 0$, involves the continuity of $dv/dy + dw/dz$; but if $k + \frac{4}{3}n = 0$, the continuity of the normal stress P requires that

$$2n\left(\frac{dv}{dy} + \frac{dw}{dz}\right) = 2n'\left(\frac{dv'}{dy} + \frac{dw'}{dz}\right),$$

when $x = 0$; and this requires that $n = n'$.

269. We must now find an expression for the mean energy per unit of volume.

The component displacements are

$$u = S\lambda, \quad v = S\mu, \quad w = S\nu,$$

where

$$S = \mathfrak{A} \cos\frac{2\pi}{V\tau}(lx + my + nz - Vt).$$

Hence

$$T = \frac{2\pi^2\mathfrak{A}^2}{\tau^2}(\rho_x\lambda^2 + \rho_y\mu^2 + \rho_z\nu^2)\cos^2\frac{2\pi}{V\tau}(lx + my + nz - Vt).$$

Since $\rho_x = B/a^2$, &c., where B denotes the rigidity; the mean kinetic energy per unit of volume is

$$T = \frac{\pi^2 \mathfrak{A} B^2}{\tau^2}\left(\frac{\lambda^2}{a^2} + \frac{\mu^2}{b^2} + \frac{\nu^2}{c^2}\right.$$

$$= \frac{\pi^2 \mathfrak{A}^2 B}{V^2 \tau^2} \cos^2 \chi$$

by (20).

The mean potential energy per unit of volume is

$$\frac{\pi^2 \mathfrak{A}^2 B}{V^2 \tau^2} \{(m\nu - n\mu)^2 + (n\lambda - l\nu)^2 + (l\mu - m\lambda)^2\}.$$

The quantity in brackets is equal to the square of the sine of the angle between the directions of propagation and vibration, and is therefore equal to $\cos^2 \chi$; whence the mean potential energy is

$$\frac{\pi^2 \mathfrak{A}^2 B}{V^2 \tau^2} \cos^2 \chi,$$

and is therefore equal to the mean kinetic energy. The mean energy is therefore

$$\frac{2\pi^2 \mathfrak{A}^2 B}{V^2 \tau^2} \cos^2 \chi.$$

Crystalline Reflection and Refraction.

270. Having discussed the preceding theory, which is due to the combined efforts of Lord Rayleigh, Sir W. Thomson and Glazebrook, we shall now consider its application to the problem of reflection and refraction at the surface of a crystal[1].

Let i be the angle of incidence; r_1, r_2 the angles which the directions of the two refracted waves make with the normal to the reflecting surface; χ_1, χ_2 the angles between the two refracted rays, and the corresponding wave normals.

The conditions at the surface of separation are

$$u = u_1, \quad v = v_1, \quad w = w_1 \text{.................}(21),$$

$$(m+n)\frac{du}{dx} + (m-n)\left(\frac{dv}{dy} + \frac{dw}{dz}\right) = (m'+n)\frac{du_1}{dx} + (m'-n)\left(\frac{dv_1}{dy} + \frac{dw_1}{dz}\right)$$
$$\text{...........}(22),$$

[1] *Proc. Lond. Math. Soc.*, Vol. xx. p. 351.

$$\frac{dv}{dx}+\frac{du}{dy}=\frac{dv_1}{dx}+\frac{du_1}{dy} \quad\text{..........................(23)}$$

$$\frac{du}{dz}+\frac{dw}{dx}=\frac{du_1}{dz}+\frac{dw_1}{dx} \quad\text{......................(24),}$$

in which $m+n$ and $m'+n$ are ultimately zero. If, therefore, we disregarded the dilatational waves altogether, we should have six equations to determine four unknown quantities. We must therefore introduce a dilatational reflected and a quasi-dilatational refracted wave, which must be eliminated, and we shall thus obtain the correct equations for determining the amplitudes of the reflected and the two refracted optical waves, and the deviation of the plane of polarization of the former.

Let the displacements in the four optical waves be

$$\Sigma = A\epsilon^{\iota\kappa(lx+my-Vt)},$$

$$\Sigma' = A'\epsilon^{\iota\kappa(-lx+my-Vt)},$$

$$\Sigma_1 = A_1\epsilon^{\iota\kappa_1(l_1x+m_1y-V_1t)},$$

$$\Sigma_2 = A_2\epsilon^{\iota\kappa_2(l_2x+m_2y-V_2t)},$$

and let the displacements in the dilatational reflected and quasi-dilatational refracted waves be

$$\mathfrak{S} = B\epsilon^{\iota\kappa(\gamma x+my-Vt)},$$

$$\mathfrak{S}_1 = B_1\epsilon^{\iota\kappa(\gamma_1 x+my-Vt)};$$

also, let I and R be the angles which the normals to these waves make with the axis of x. Let θ, θ' be the angles which the directions of vibration in the incident and reflected optical waves make with the axis of z; θ_1, θ_2 the angles which the projections upon their respective waye-fronts of the directions of vibration in the two refracted waves, make with this axis.

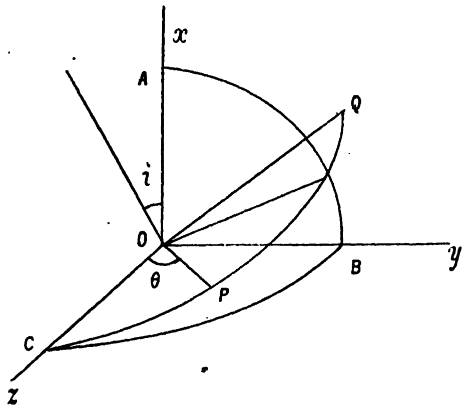
Then, omitting the common exponential factor, and also all terms involving Σ_2, which are of the same form as those involving Σ_1, and can therefore be supplied at the end of the investigation, we have at the surface

$$\left.\begin{aligned} u &= A\cos AP + A'\cos AP' + B\cos I \\ v &= A\cos BP + A'\cos BP' + B\sin I \\ w &= A\cos CP + A'\cos CP' \end{aligned}\right\} \quad\text{.........(25)}$$

for the first medium.

For the second medium

$$\begin{aligned} u_1 &= A_1(\cos\chi_1 \cos AP_1 - \sin\chi_1 \cos r_1) - B_1 \cos R \\ v_1 &= A_1(\cos\chi_1 \cos BP_1 + \sin\chi_1 \sin r_1) + B_1 \sin R \quad \ldots(26). \\ w_1 &= A_1 \cos\chi_1 \cos CP_1 \end{aligned}$$



Since $du/dy = du_1/dy$ when $x = 0$, and $du/dz = du_1/dz = 0$ (23) and (24) give

$$(A\cos BP - A'\cos BP')\,\kappa l + B\kappa\gamma \sin I =$$
$$A_1(\cos\chi_1 \cos BP_1 + \sin\chi_1 \sin r_1)\kappa_1 l_1 + B_1\kappa\gamma_1 \sin R \ldots\ldots\ldots (27),$$

$$(A\cos CP - A'\cos CP')\,\kappa l = A_1\kappa_1 l_1 \cos\chi_1 \cos CP_1 \ldots\ldots\ldots\ldots\ldots(28).$$

Since $m + n$ and $m' + n$ are ultimately zero, and $dv/dy = dv_1/dy$, both sides of (22) ultimately become identically equal, and this equation need not therefore be considered.

Now, if Λ, Λ_1 be the wave-lengths of the waves $\mathfrak{S}$, $\mathfrak{S}_1$; U, U_1 their velocities of propagation,

$$\kappa\gamma = \frac{2\pi}{\Lambda}\cos I, \quad \kappa\gamma_1 = -\frac{2\pi}{\Lambda_1}\cos R,$$

$$\kappa m = \frac{2\pi}{\Lambda}\sin I = \frac{2\pi}{\Lambda_1}\sin R = \frac{2\pi}{\lambda}\sin i = \&c.,$$

$$\kappa V = \frac{2\pi}{\Lambda}U = \frac{2\pi}{\Lambda_1}U_1;$$

and therefore, since U, U_1 are ultimately zero, Λ, Λ_1 are also ultimately zero; whence $I = 0$, $R = 0$, and therefore γ, γ_1 are ultimately infinite. Also,

$$\kappa\gamma \sin I = \frac{2\pi}{\Lambda}\cos I \sin I = \kappa m \cos I.$$

Writing out the equation $u = u_1$ in full, multiplying by κm, and subtracting from (27), we obtain

$$(A \cos BP - A' \cos BP')\,\kappa l - (A \cos AP + A' \cos AP')\,\kappa m$$
$$= A_1 (\cos \chi_1 \cos BP_1 + \sin \chi_1 \sin r_1)\,\kappa_1 l_1 - A_1 (\cos \chi_1 \cos AP_1 - \sin \chi_1 \sin r_1)\,\kappa m \ldots\ldots\ldots (29).$$

From the preceding investigation we see, that B and B_1 are not zero, but finite, and therefore the existence of the waves $\mathfrak{S}$, $\mathfrak{S}_1$ cannot be entirely ignored; but, since $I = R = 0$ the terms involving B, B_1 disappear from the equation $v = v'$, which gives

$$A \cos BP + A' \cos BP' = A_1 (\cos \chi_1 \cos BP_1 + \sin \chi_1 \sin r_1) \ldots (30),$$

and the equation $w = w'$ gives

$$A \cos CP + A' \cos CP' = A_1 \cos \chi_1 \cos CP_1 \ldots\ldots (31).$$

Equations (28), (29), (30), and (31) contain the complete solution of the problem.

Now

$$\cos AP = \sin i \sin \theta, \quad \cos BP = \cos i \sin \theta, \quad \cos CP = \cos \theta,$$

with similar expressions for $\cos AP_1$, &c.; also,

$$\cos AP' = \sin i \sin \theta', \quad \cos BP' = -\cos i \sin \theta', \quad \cos CP' = \cos \theta',$$

whence (28), (29), (30) and (31) become

$$\begin{aligned}
(A \cos \theta - A' \cos \theta') \cot i &= A_1 \cot r_1 \cos \chi_1 \cos \theta_1 \\
(A \sin \theta + A' \sin \theta') \operatorname{cosec} i &= A_1 \operatorname{cosec} r_1 \cos \chi_1 \sin \theta_1 \\
(A \sin \theta - A' \sin \theta') \cos i &= A_1 (\cos r_1 \cos \chi_1 \sin \theta_1 + \sin r_1 \sin \chi_1) \\
A \cos \theta + A' \cos \theta' &= A_1 \cos \chi_1 \cos \theta_1
\end{aligned} \qquad \ldots\ldots\ldots\ldots\ldots (32),$$

in which equations we are to recollect, that we are to add to the right-hand sides terms in A_2 similar to those involving A_1.

The preceding equations may also be obtained by a process, which does not involve the introduction of the dilatational waves.

Since the continuity of u, v, w involves the continuity of their differential coefficients with respect to y and z, the first of (21) together with (23) and (24) involve the continuity of the rotations ζ and η; also, since $m = m' = -n$, both sides of (22) are identically equal, and therefore this equation disappears; we are thus left with the last two of (21). The surface conditions are therefore

$$v = v_1, \quad w = w_1;$$
$$\eta = \eta_1, \quad \zeta = \zeta_1;$$

which furnish four equations to determine the four unknown quantities.

Equations (32) determine the amplitudes of the reflected and refracted waves, but according to § 10, the intensity is to be measured by the mean energy per unit of volume. Accordingly by § 269, if I, I', I_1, I_2 denote the square roots of the intensities of the four waves

$$\frac{A}{I\sin i}=\frac{A'}{I'\sin i}=\frac{A_1\cos\chi_1}{I_1\sin r_1}=\frac{A_2\cos\chi_2}{I_2\sin r_2},$$

whence (32) become

$$\left.\begin{aligned}(I\cos\theta+I'\cos\theta')\sin i&=I_1\cos\theta_1\sin r_1+I_2\cos\theta_2\sin r_2\\(I\cos\theta-I'\cos\theta')\cos i&=I_1\cos\theta_1\cos r_1+I_2\cos\theta_2\cos r_2\\I\sin\theta+I'\sin\theta'&=I_1\sin\theta_1+I_2\sin\theta_2\\(I\sin\theta-I'\sin\theta')\sin 2i&=I_1(\sin\theta_1\sin 2r_1+2\sin^2 r_1\tan\chi_1)\\&\quad+I_2(\sin\theta_2\sin 2r_2+2\sin^2 r_2\tan\chi_2)\end{aligned}\right\}\ldots(33).$$

271. We shall hereafter show, that these equations are exactly the same as those furnished by the electromagnetic theory, and we shall postpone the complete discussion of them, until we deal with that theory; but it will be desirable to consider the results to which they lead, when both media are isotropic.

1st. Let the light be polarized in the plane of incidence; then $\theta=\theta'=\theta_1=0$; $\chi_1=\chi_2=0$, and $I_2=0$, whence

$$(I+I')\sin i=I_1\sin r,$$
$$(I-I')\cos i=I_1\cos r,$$

from which we deduce

$$I'=-I\frac{\sin(i-r)}{\sin(i+r)},$$
$$I_1=\frac{I\sin 2i}{\sin(i+r)},$$

which are the same formulæ as those obtained by Fresnel.

2nd. Let the light be polarized perpendicularly to the plane of incidence; then $\theta=\theta'=\theta_1=\frac{1}{2}\pi$, and

$$I+I'=I_1,$$
$$(I-I')\sin 2i=I_1\sin 2r,$$

whence

$$I'=I\frac{\tan(i-r)}{\tan(i+r)},$$
$$I_1=\frac{I\sin 2i}{\sin(i+r)\cos(i-r)},$$

which are Fresnel's formulæ for light polarized perpendicularly to the plane of incidence. See §§ 172 and 175.

272. The principal results of this theory are, (i) that it leads to a wave-surface, which is approximately though not accurately Fresnel's wave-surface, unless k is absolutely and not approximately equal to $-\frac{1}{3}n$; (ii) that although the direction of vibration within the crystal is not the same as in Fresnel's theory (being perpendicular to the ray instead of to the wave-normal), yet it makes the vibrations of polarized light on emerging from the crystal, perpendicular to the plane of polarization; (iii) the equations, which determine the intensities in the case of crystalline reflection and refraction are, as we shall hereafter see, identical with those which are furnished by the electromagnetic theory; and when both media are isotropic, the results agree with those obtained by Fresnel. Also, as soon as the assumptions have been made, that k is equal, or nearly so to $-\frac{1}{3}n$, and that double refraction arises from the circumstance, that crystalline media behave as if they were æolotropic as regards density; results which can be proved to be very approximately true, are capable of being deduced without the aid of any of those additional assumptions, which in many cases are indispensable in order to obtain a particular analytical result.

Theory of Rotatory Polarization.

273. When bodily forces act upon the ether, the equations of motion will be of the form

$$\rho_x \frac{d^2u}{dt^2} = (A - B)\frac{d\delta}{dx} + B\nabla^2 u + X \text{ ....(34).}$$

&c. &c.

Now the photogyric properties of quartz and turpentine must be due to the peculiar molecular structure of such substances; and we may endeavour to construct a theory of rotatory polarization, by supposing that the effect of the mutual reaction of ether and matter modifies the motion of ethereal waves in a peculiar manner, which may be represented by the introduction of certain bodily forces. The mathematical form of these forces is a question of speculation, but we shall now, following MacCullagh[1], show that

[1] *Trans. Roy. Irish Acad.* Vol. XVII. p. 461.

rotatory polarization may be accounted for by supposing that these forces are of the form

$$X = p_3 \frac{d^3v}{dz^3} - p_2 \frac{d^3w}{dy^3}, \quad Y = p_1 \frac{d^3w}{dx^3} - p_3 \frac{d^3u}{dz^3}, \quad Z = p_2 \frac{d^3u}{dy^3} - p_1 \frac{d^3v}{dx^3} \quad (35).$$

274. For an isotropic medium such as syrup or turpentine, $\rho_x = \rho_y = \rho_z$; $p_1 = p_2 = p_3$. If therefore the axis of z be taken as the direction of propagation, u and v will be functions of z alone; whence (34) combined with (35) reduce to

$$\left.\begin{aligned} \rho \frac{d^2u}{dt^2} &= n \frac{d^2u}{dz^2} + p_3 \frac{d^3v}{dz^3} \\ \rho \frac{d^2v}{dt^2} &= n \frac{d^2v}{dz^2} - p_3 \frac{d^3u}{dz^3} \end{aligned}\right\} \quad \text{.................(36).}$$

To solve these equations, assume

$$u = L\epsilon^{\frac{2\iota\pi}{\tau}\left(\frac{z}{V} - t\right)}, \quad v = M\epsilon^{\frac{2\iota\pi}{\tau}\left(\frac{z}{V} - t\right)} \quad \text{...........(37),}$$

where L and M may be complex constants, τ is the period, and V the velocity of propagation. Substituting in (36), and putting $U^2 = n/\rho$, $p_3/\rho = p$, we obtain

$$(V^2 - U^2)L - (2\iota\pi p/V\tau)M = 0,$$
$$(V^2 - U^2)M + (2\iota\pi p/V\tau)L = 0.$$

Now the rotatory effect, which depends on p, is very small, whence in the terms involving p, U may be written for V; we accordingly obtain

$$M = \mp \iota L,$$
$$V^2 = U^2 \pm 2\pi p/U\tau \quad \text{....................(38).}$$

If the incident light be represented by

$$u = L\cos 2\pi t/\tau, \quad v = 0,$$

and if V_1, V_2 denote the greater and lesser values of V, we shall obtain in real quantities

$$u = \tfrac{1}{2}L\cos\frac{2\pi}{\tau}\left(\frac{z}{V_1} - t\right) + \tfrac{1}{2}L\cos\frac{2\pi}{\tau}\left(\frac{z}{V_2} - t\right),$$
$$v = \tfrac{1}{2}L\sin\frac{2\pi}{\tau}\left(\frac{z}{V_1} - t\right) - \tfrac{1}{2}L\sin\frac{2\pi}{\tau}\left(\frac{z}{V_2} - t\right).$$

Accordingly

$$u = L\cos\frac{2\pi}{\tau}\left\{\tfrac{1}{2}z\left(\frac{1}{V_1} + \frac{1}{V_2}\right) - t\right\}\cos\frac{\pi z}{\tau}\left(\frac{1}{V_2} - \frac{1}{V_1}\right),$$
$$v = -L\cos\frac{2\pi}{\tau}\left\{\tfrac{1}{2}z\left(\frac{1}{V_1} + \frac{1}{V_2}\right) - t\right\}\sin\frac{\pi z}{\tau}\left(\frac{1}{V_2} - \frac{1}{V_1}\right.$$

Whence if ψ be the angle through which the plane of polarization is rotated, *measured towards the right hand of a person who is looking along the ray*

$$\tan\psi = -v/u = \tan\frac{\pi z}{\tau}\left(\frac{1}{V_2} - \frac{1}{V_1}\right).$$

Substituting the values of V_2, V_1 from (38) we obtain approximately

$$\psi = \frac{2\pi^2 pz}{U^4\tau^2} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(39),$$

which is the expression for Biot's law. The sign of p is positive or negative, according as the medium is right-handed or left-handed.

If the wave were travelling in the opposite direction, the sign of t would have to be reversed in (37); this would not make any alteration in the form of (39), but since in this case z would be negative, the rotation would be in the opposite direction. This agrees with experiment.

275. We must now consider the theory of quartz. Taking the axis of the crystal as the axis of z, we must put

$$\rho_x = \rho_y, \quad p_1 = p_2 = 0;$$

whence writing

$$n/\rho_x = a^2, \quad n/\rho_z = c^2, \quad q = p_1/n,$$

and recollecting that $A - B = -n$, the equations of motion become

$$\left.\begin{aligned} \frac{d^2u}{dt^2} &= a^2\left(\nabla^2 u - \frac{d\delta}{dx}\right) + qa^2\frac{d^3v}{dz^3} \\ \frac{d^2v}{dt^2} &= a^2\left(\nabla^2 v - \frac{d\delta}{dy}\right) - qa^2\frac{d^3u}{dz^3} \\ \frac{d^2w}{dt^2} &= c^2\left(\nabla^2 w - \frac{d\delta}{dz}\right) \end{aligned}\right\} \quad\ldots(40).$$

Since everything is symmetrical with respect to the axis, we may without loss of generality suppose, that the axis of y is parallel to the wave-front; we may therefore put

$$u = A_1 S, \quad v = A_2 S, \quad w = A_3 S,$$

where $$S = \epsilon^{2\iota\pi/V\tau\,.\,(lx+nz-Vt)}.$$

Substituting in (40) we obtain

$$(V^2 - a^2n^2)A_1 + a^2lnA_3 - (2\iota\pi/V\tau)\,qa^2n^3A_2 = 0,$$
$$(V^2 - a^2)A_2 + (2\iota\pi/V\tau)\,qa^2n^3A_1 = 0,$$
$$(V^2 - c^2l^2)A_3 + c^2lnA_1 = 0,$$

whence eliminating A_1, A_2, A_3, and putting θ for the angle which the direction of propagation makes with the axis of the crystal, we obtain

$$(V^2 - a^2)(V^2 - a^2\cos^2\theta - c^2\sin^2\theta)$$
$$= (4\pi^2/V^4\tau^2)\,q^2a^4\cos^6\theta\,(V^2 - c^2\sin^2\theta) \;......(41).$$

This equation is a biquadratic for determining V^2, but since q is a small quantity, we may put U for V in the right-hand side; and we thus obtain a quadratic equation for determining the two values of V^2.

It follows from (41), that when the direction of propagation is perpendicular to the axis, so that $\theta = \frac{1}{2}\pi$, quartz behaves in the same way as Iceland spar. When the direction of propagation is parallel to the axis, so that $\theta = 0$, the first two of (40) are of the same form as (36); consequently there are two waves, which travel through the crystal with different velocities, and are circularly polarized in opposite directions. For directions which are slightly inclined to the axis, the two waves are elliptically polarized in opposite directions; but owing to the smallness of q, it follows that the right-hand side of (41) diminishes rapidly as θ increases, so that when θ is not small, the elliptic polarization becomes insensible, and the two waves are sensibly plane polarized.

CHAPTER XVI.

MISCELLANEOUS EXPERIMENTAL PHENOMENA.

276. THE present chapter is descriptive and not mathematical; and its object is to give an account of a variety of experimental phenomena. The subjects of which it treats are, Dispersion, Spectrum Analysis, Absorption, Colours of Natural Bodies, Dichromatism, Anomalous Dispersion, Selective Reflection, Fluorescence, Calorescence, and Phosphorescence. We shall give an account of the principal experimental results connected with these phenomena, so far as they relate to the Theory of Light; and in the next chapter, we shall enquire how far they may be explained by theoretical considerations.

Dispersion.

277. When sunlight, proceeding from a horizontal slit, is refracted by a prism whose vertex is downwards, and the emergent light is received upon a screen, it is found that the image of the slit, instead of consisting of a narrow line of white light, presents the appearance of a coloured band, which is called the *solar spectrum*. The order of succession of the colours is violet, indigo, blue, green, yellow, orange, red; the red end of the band being lowest, and the violet highest. This experiment shows, that light of different colours possesses different degrees of refrangibility, violet being the most refrangible, and red the least. All transparent bodies, which are capable of resolving light into its constituent colours, are called *dispersive* media; and the phenomenon of resolution is called *dispersion.*

278. Observations of eclipses have shown, that the velocity of light of all colours is the same in vacuo. We have also proved by more than one theory, that the index of refraction of light, which passes from a vacuum into a transparent medium, is equal to the ratio of the velocity of light in vacuo, to the velocity of light in the transparent medium; hence the velocity of light in a dispersive medium is a function of the period, and is greater for red light than for violet. Now, subject to an exception, which will be explained in § 329, it follows from the fundamental principles of Dynamics, that the period of light of any particular colour is an invariable quantity, which is independent of the physical constitution of the medium through which the light is passing; hence the wave-length of light of any particular colour, depends upon the constitution of the medium, and is smaller in glass than in vacuo. It can also be proved experimentally, that the wave-length of violet light in vacuo is less than the wave-length of red, and that the wave-lengths of different colours increase continuously in going down the spectrum from violet to red; whence the period of violet light is less than that of red, and the periods increase continuously in going down the spectrum.

279. That the colour of light depends upon the period, is a fact of fundamental importance in Physical Optics; and when we consider some of the dynamical theories which have been proposed to explain dispersion, we shall see that there are strong grounds for thinking, that the qualities of transparency and opacity do not depend so much upon the particular substance of which the medium consists, as upon the period of the ethereal waves.

280. A table of the principal wave-lengths and periods is given on p. 284, but at present it will be only necessary to notice, that the *frequency* (or number of vibrations per second) of the extreme red waves is about 395×10^{12}, and the frequency of the extreme violet is about 763×10^{12}. Accordingly the interval between the extreme red and violet is slightly less than 2, and the sensitiveness of the eye is therefore confined to less than an octave; on the other hand the sensitiveness of the ear to sound extends to about eleven octaves.

Spectrum Analysis.

281. Dispersion was discovered by Newton[1] in 1675; but in 1802 Wollaston[2] observed, that if sunlight is allowed to pass through a very narrow slit, the spectrum is not continuous, but is crossed by a number of fine dark lines. The investigation of these dark lines was subsequently taken up by Frauenhofer[3] in 1814, who observed 576 of them, and they are now universally known by his name. Frauenhofer selected certain of the principal lines as land-marks, which he denoted by the letters A, a, B, C, D, E, b, F, G, H, I, and which we shall proceed to consider.

282. A is a line in the extreme red portion of the spectrum; a is a group of lines, and B and C are well defined lines also in the red; D is a double line in the orange, which consists of two lines very close together; E lies in the yellow end of the green; b is a group of lines in the green; F is a line in the blue end of green; G lies in the indigo; H in the violet; and I is the extreme end of the spectrum. The wave-lengths of these lines were first measured by Frauenhofer, subsequently by Ångström[4], and more recently by Rowland, Bell[5] and others in America, and some of the values obtained are given in the following table in tenth-metres. A tenth-metre is 10^{-10} of a metre.

Line	Ångström	Bell
A	7604·00	7621·31
B	6867·00	6884·11
C	6562·01	6563·07
D_1	5895·00	5896·18
D_2	5889·04	5890·22
E_1	5269·13	5270·52
E_2		5269·84
F	4860·72	4861·51
G	4307·25	...
H_1	3968·01	...
H_2	3933·00	...

Frauenhofer's examination of these dark lines is one of the most important scientific events of the present century. It has laid the

[1] *Opticks*, Book I. Part I.
[2] *Phil. Trans.* 1802, p. 378.
[3] *Denkschriften der Münchener Akad.* 1814.
[4] *Recherches sur le Spectre Solaire, Spectre Normal du Soleil*, p. 25. Upsala, 1868.
[5] *Phil. Mag.* (5) vol. xxv. p. 368.

foundations of the modern sciences of Spectrum Analysis and of Solar and Stellar Chemistry, and has led to the discovery, that numerous elements which exist in the earth, also exist in the sun and fixed stars.

283. Spectrum analysis may be considered to have commenced by the discovery of Bunsen and Kirchhoff[1] in 1859, that every chemical substance, when incandescent, produces its own particular spectrum. Thus if any salt of sodium is burnt in a spirit lamp, the spectrum, instead of being continuous, consists of two bright yellow lines, which occupy the same position as the *D* lines in the solar spectrum. Potassium gives two red lines, which coincide with the lines *A* and *B*, and a violet line between *G* and *H*. Lithium gives a strong red line between *B* and *C*, and three fainter lines in the orange, green and greenish blue. Sodium, potassium and lithium accordingly colour a flame yellow, violet and red respectively. Strontium also colours a flame red, but its spectrum can never be mistaken for that of lithium, since the spectrum of the former consists of a number of lines in the red and orange, and a single line in the blue. The spectrum of barium consists of a number of lines in the yellow and green, and this substance colours a flame green. Gases also, when incandescent, exhibit spectra. That of hydrogen consists of a red, blue-green and an indigo line, each of which respectively coincides with the lines *C*, *F* and *G*; whilst the spectra of oxygen and of nitrogen are more complicated.

284. Spectrum analysis furnishes an exceedingly delicate test of the presence or absence of any element in any chemical compound; for if we place a small portion of the compound in the flame of a spirit lamp, the presence of the element will be detected by its spectrum. It has by this means been discovered, that sodium is one of the most widely distributed substances; also lithium, which was formerly supposed to be rare, is found to exist in numerous minerals, and also in plants and in the bodies of animals. Spectrum analysis has also led to the discovery of new elements; for if the spectrum of a substance shows lines, which do not correspond to the lines of any known element, the obvious inference is, that these lines are due to the presence of some previously unknown element. It was in this way, that Bunsen

[1] Chemical Analysis by Spectrum Observations; translated, *Phil. Mag.* Aug. 1860, p. 89; Nov. 1861, p. 329; Dec. 1861, p. 498.

discovered two new alkaline metals in the mineral springs of Baden-Baden and of Dürkheim, which he named *cæsium* and *rubidium*. The spectrum of the former consists of two lines in the blue between F and G, and a number of lines between B and E; whilst the spectrum of the latter consists of two lines in the red below A, two lines in the violet a little above G, and a number of lines between C and F. Thallium, which was discovered by Crookes[1] in 1861, is distinguished by a green line a little below E. The spectrum of indium, which was discovered by Reich and Richter in 1864, consists of a blue line and an indigo line; whilst that of gallium, which was discovered by Lecoq de Boisbaudan in 1875, consists of two bright violet lines.

The methods of spectrum analysis afford so delicate a test of the presence of an element, that 3×10^{-6} of a millegramme of sodium, 10^{-5} of a millegramme of lithium, and 6×10^{-5} of a millegramme of strontium can be detected.

285. Up to the present time, we have confined our attention to the visible portion of the spectrum; we must now show that ethereal waves exist, whose periods are greater than those of the extreme red waves, and less than those of the extreme violet. These portions of the spectrum are called the infra-red and the ultra-violet respectively.

286. The infra-red waves are waves of dark heat, and were first discovered by Sir W. Herschel in 1800 by means of their thermal effects. He found, that when a thermometer was held for a short time in that portion of the spectrum, which lies below the visible red, an increase of temperature was observed; but perhaps the most striking way of showing the existence of the infra-red waves, is by means of the following experiment due to Tyndall. A solution of iodine in disulphide of carbon is opaque to the luminous waves, but is transparent to the infra-red waves; accordingly if sunlight, or the light from an electric lamp, is passed through such a solution, the infra-red waves are alone transmitted. If therefore the rays are now brought to a focus by means of a convex lens, their thermal effects can be exhibited by placing at the focus a little gunpowder, which immediately explodes, or by lighting a cigar. Tyndall also found that the amount of radiation, which passed through the solution, was $\frac{7}{8}$ths of the whole[2].

[1] *Phil. Mag.* vol. XXI. (1861), p. 301.

[2] Tyndall, *Phil. Trans.* 1870, p. 333.

287. The ultra-violet or actinic waves are noted for the chemical effects which they produce. This can be shown by passing the light from an electric lamp, or from burning magnesium, which contains a large amount of ultra-violet light, through any mixture which absorbs these waves, and then allowing the light to fall upon a photographic plate, when it will be found that very little effect has been produced. Hence it is necessary to develop photographs in a room with yellow blinds, which are opaque to these waves.

288. It was formerly supposed, that the infra-red portion of the spectrum produced the maximum heating effect; that the yellow was the most luminous portion; whilst the maximum chemical effect was contained in the violet and ultra-violet portions. More recent investigations have shown, that this view is not correct, and that every portion of the spectrum exhibits all three effects. If two spectra of equal lengths, formed by a prism and by a diffraction grating, be examined, it will be found that in the spectrum formed by the prism, the maximum heating effect is in the red, whilst in the spectrum formed by the grating, it is in the yellow. The reason of this is, that the spectrum formed by the prism is more spread out towards the violet end and more compressed towards the red end, than the spectrum formed by the grating. This was first pointed out by Draper. The distribution of heat in the spectrum has been investigated by Prof. Langley, by means of an instrument invented by himself and called a bolometer, which is of so delicate a character, that a difference of temperature amounting to 10^{-4} of a degree centigrade can easily be noted. By means of this instrument, Langley detected the heating effects of the ultra-violet waves, although the radiation was so weak, that if it fell uninterruptedly for over 1000 years upon a kilogramme of ice, the latter would not be entirely melted.

289. The actinic effects of the infra-red waves have been investigated by Captain Abney, who has succeeded in preparing sensitive photographic films capable of being affected by them.

290. The researches of Langley, Abney and others have greatly extended our knowledge of the solar spectrum, and have shown that instead of being confined to rather less than an octave, the portion hitherto examined extends from about wave-lengths 1400 to 18000 tenth-metres, that is to about four octaves.

291. Having discussed the solar spectrum, and explained how spectrum analysis enables us to detect the presence of chemical elements which exist in the earth, and can therefore be experimented upon; we must now enquire, how the presence or absence of these elements in the sun and fixed stars, can also be ascertained.

292. Frauenhofer observed, that the bright yellow line of the spectrum of sodium coincided with the line D of the solar spectrum, and he suggested that the dark lines of the latter were due to some agency, which lay outside the earth's atmosphere. Brewster, who was acquainted with Frauenhofer's researches, observed that the two red lines in the spectrum of potassium coincided with the lines A and B, and was struck with the coincidence; but neither Frauenhofer nor Brewster appear to have had any idea of the true cause. This seems to have been first pointed out by Sir G. Stokes[1] about 1852, and depends upon the following considerations.

It is a well-known dynamical theorem, due to the late Sir J. Herschel, that when a material system is acted upon by a periodic force, whose period is equal or nearly equal to one of the free periods of the system, the amplitude of the corresponding forced vibration will be large. Stokes accordingly suggested, that sodium by virtue of its molecular structure was capable of vibrating in the periods of the light, which corresponds to the two D lines of the solar spectrum. If therefore light from a sodium flame be passed through a stratum of vapour of sodium, the molecules of the latter will be thrown into a state of vibration, and owing to the coincidence of the free and forced periods, the amplitudes of the vibrations of the molecules of sodium vapour will be large. The sodium vapour will thus become the seat of energy, and since this energy is necessarily supplied by the energy of the waves of sodium light, which are incident upon the stratum, the amount of energy in the form of waves of light which emerges from the stratum, will be very much less than that which entered; and accordingly if the stratum be sufficiently thick, no energy will emerge, and light will be absorbed. It appears that in 1849 Foucault[2] experimentally proved, that light from a sodium flame was absorbed by sodium vapour, but the experiment attracted no

[1] *Phil. Mag.* vol. xx. (1860), p. 20.
[2] *Journal de l'Institut*, Feb. 7th, 1849.

attention, and the subject was not properly investigated until it was taken up by Kirchhoff in 1859.

293. Kirchhoff first examined the spectrum of incandescent sodium vapour, and found that the two bright lines of its spectrum coincided with the two D lines of the solar spectrum. He also examined the spectra of calcium, magnesium, iron and a variety of other substances, and found that the bright lines of their spectra coincided with certain definite dark lines of the solar spectrum. From these observations he deduced the following two very important laws:—

I. *Whenever a group of dark lines in the solar spectrum coincides with the bright lines in the spectrum of any incandescent substance, that substance is present in the sun.*

II. *A substance which emits light of definite periods when incandescent, absorbs light of the same periods as it would emit when incandescent.*

294. Ångström and Thalen shortly afterwards showed, that hydrogen was present in the sun, and Norman Lockyer has added lead, potassium and a variety of other substances to the list.

The spectra of many of the fixed stars have also been examined; and it has been found that hydrogen exists in Sirius, and also in the Nebulæ.

295. The physical theory suggested by Stokes, coupled with the experiments of Kirchhoff[1], furnish a satisfactory explanation of these phenomena. The sun possesses an incandescent gaseous atmosphere, which surrounds a solid nucleus having a still higher temperature. If we could examine the spectrum of the solar atmosphere, we should observe bright bands, which correspond to the substances contained in it; but the more intense luminosity of the solid nucleus does not permit the spectrum of the sun's atmosphere to appear. The solar atmosphere accordingly absorbs the light, which the gases it contains would emit, and consequently dark bands appear in the solar spectrum, which correspond to the light absorbed. If any particular substance, such as sodium, is present in the sun's nucleus, the sun's atmosphere will also contain

[1] *Phil. Mag.* vol. XIX. (1860), p. 193; vol. XX. (1860), p. 1; vol. XXI. (1861), p. 185. See also, Kirchhoff and Bunsen, *Phil. Mag.* XX. (1860), p. 89; Kirchhoff, *Gesammelte Abhandlungen*, p. 598.

this substance in the gaseous state, and the light emitted from the sodium in the nucleus, will be wholly or partially absorbed by the sodium gas contained in the atmosphere, and the dark lines D will be produced in the solar spectrum.

296. One of the most interesting astronomical applications of spectrum analysis, is the determination of the relative motions of the sun and the fixed stars. It is well-known, that the sun is moving at the present time in the direction of the constellation of Hercules. Now if a fixed star is observed to have a *proper motion*, it is evident on account of the enormous distances between the sun and stars, that the directions of motion of the sun and star must be nearly at right angles to the line joining them; but if the directions of motion are nearly parallel to this line, astronomical methods furnish no means of determining, whether the two bodies are approaching towards or receding from one another. It is at this point that spectrum analysis comes to our aid, and enables the direction of the relative motion to be determined, by means of the shifting of the fixed lines of the spectrum.

297. In order to understand how this is effected, we must explain a theorem originally due to Doppler, which is usually known as Doppler's Principle[1].

Let us suppose, that a source S of light is emitting spherical waves, whose velocity of propagation is V; and that the source and the observer are in motion. Let the source be reduced to rest, by impressing upon the observer a velocity equal and opposite to that of the source; and let u be the component of the relative velocity of the observer *towards* the source, along the line joining him with the latter. Since the observer is unconscious of his own motion, the waves emitted by the source, will appear to be travelling with a velocity $V+u$; hence if λ be the wave-length, and τ' the apparent period, that is the time which elapses between the passage of each successive crest of the waves,

$$(V+u)\,\tau' = \lambda.$$

But if τ be the actual period,

$$V\tau = \lambda,$$

whence
$$\tau' = \frac{V\tau}{V+u}.$$

[1] *Böhm. Gesell. Abh.* II. 1841—2, p. 485.

Now the position of any line in the spectrum depends upon the number of waves, which fall upon the retina in a second. If therefore the observer be in motion, the position of any line will depend upon the *apparent* period, instead of upon the actual period. Accordingly if the observer is moving *towards* the star, so that u is positive, the apparent period will be less than the actual period, and any fixed line will be shifted towards the violet end of the spectrum. The converse is the case, when the observer is moving away from the star, so that u is negative, and the fixed lines are shifted towards the red end of the spectrum.

298. By the aid of this method Dr Huggins[1] ascertained in 1868, that in the spectrum of Sirius, the hydrogen line F is slightly shifted towards the red end of the spectrum; from which it follows, that there is a motion of recession between the earth and Sirius, and he calculated that the relative velocity is about twenty-nine miles per second.

299. Further observations made by Huggins[2] with improved instruments, not only confirmed his previous observations upon Sirius, but showed that many other stars are in motion, and that some are moving towards, and others from the earth. Thus Sirius, Rigel, Castor, Regulus and δ Ursæ Majoris, which are situated in that part of the heavens which is opposite to Hercules, are moving from the earth; whilst Arcturus, Vega, and α Cygni, which are situated in the neighbourhood of this constellation, are moving towards the earth.

300. The outer portion of the solar atmosphere consists chiefly of white-hot hydrogen gas, which is constantly agitated by storms of the most violent character; and by observing the peculiar alterations in the breadth and position of the hydrogen line F, Norman Lockyer has been enabled to calculate the velocity with which these masses of gas are moving.

The reader who wishes to pursue this subject further, is recommended to consult Roscoe's *Spectrum Analysis*, and the various memoirs and treatises relating to Spectrum Analysis, and to Solar and Stellar Chemistry. We must now pass on to the subject of absorption.

[1] *Phil. Trans.* 1868.

[2] *Phil. Trans.* 1872. In this paper a table is given of the stars, which are approaching towards or receding from the earth. See also Roscoe, *Spectrum Analysis* (fourth edition), pp. 329, 355, and pp. 400—410.

Selective Absorption.

301. From the preceding discussion on spectrum analysis, it will be readily understood, that the reason why sherry appears to be of an orange colour is, that this liquid absorbs blue and green light; whilst the colour of claret is due to the fact, that it absorbs yellow and green light, and transmits red and violet. The property which certain transparent substances possess of refusing to transmit light of certain colours, whilst allowing light of other colours to pass freely through, is called *selective absorption*. Even substances which are apparently transparent to light of all colours, exhibit selective absorption when sufficiently thick. Thus ordinary glass, such as is used for window panes, appears green when the thickness is sufficiently great, thus showing that glass is not perfectly transparent, but has a tendency to absorb certain kinds of light. Air also has a tendency to absorb blue and green light, and in a less degree yellow light; for the colour of the sun at mid-day is a whitish yellow, whilst at sunset or sunrise it is red, thus showing that a sufficient thickness of air refuses a passage to all but the red rays, and this fact has led many physicists to believe that the actual colour of the sun is blue. Silver leaf, which is opaque to luminous rays, transmits the ultra-violet rays, gold leaf transmits green rays, whilst glass which is transparent to luminous rays, absorbs the actinic rays, and also the rays of dark heat[1]. Quartz, on the other hand, transmits both the luminous and the actinic rays; and on this account it is customary to use quartz prisms when experimenting upon the latter rays. It therefore appears, that transparency and opacity must not be regarded as qualities, which form part of the intrinsic properties of substances, but which depend rather upon the periods of vibration of the ethereal waves.

302. Some of the dynamical theories which have been proposed to explain absorption will be considered in the next chapter, but the following example will assist the reader to understand how it is, that a medium can be opaque to waves whose periods lie between certain limits.

The mechanical properties of a substance, which is incapable

[1] Lecher and Pernter, On the Absorption of Dark-heat Rays by Gases and Vapours, *Sitz. der K. Akad. der Wissen. in Wien*, July 1880; translated, *Phil Mag.*, Jan. 1881. In this paper references are given to the investigations of other experimentalists.

of transmitting waves whose periods lie beyond certain limits, may be illustrated by supposing that for waves travelling parallel to x, the equation of motion of the ether is

$$\frac{d^2w}{dt^2} = a^2\frac{d^2w}{dx^2} + b^2\frac{d^4w}{dx^2dt^2} \qquad \text{.................(1).}$$

If in this equation we put $b = 0$, we obtain the ordinary equation which is furnished by Green's theory; and the last term may be conceived to represent the action of the substance upon the ether. Putting

$$w = A\epsilon^{\frac{2\iota\pi}{V\tau}(x - Vt)},$$

we obtain $$V^2 = a^2 - b^2/\tau^2,$$

and therefore the index of refraction is

$$\mu^2 = \frac{a^2}{a^2 - b^2/\tau^2}.$$

When $\tau = \infty$, $\mu = 1$; as τ diminishes, μ increases but remains real as long as $\tau > b/a$. When $\tau = b/a$, $\mu = \infty$ and $V = 0$; and when $\tau < b/a$, μ^2 is negative and V is imaginary. The medium is therefore incapable of propagating waves, whose period is less than b/a; and an equation of the form (1) might therefore be employed to illustrate the action of a medium, which is opaque to the ultra-violet waves.

303. The theoretical explanation of the absorption of certain colours, depends upon the dynamical theorem to which allusion has been made. The molecules of all substances are capable of vibrating in certain definite periods, which are the free periods of the substance. The number of different free periods depends upon the molecular structure of the substance; and in all probability, the more complex the substance is, the more numerous are the free periods of the molecules. If therefore any of the free periods lie within the limits of the periods of the visible spectrum, absorption will take place. Suppose, for example, that the velocity of light in an absorbing medium were given by an equation of the form

$$\frac{A}{V^2} = \frac{1}{\kappa_1^2 - \tau^2} - \frac{1}{\kappa_2^2 - \tau^2} = \frac{\kappa_2^2 - \kappa_1^2}{(\tau^2 - \kappa_1^2)(\tau^2 - \kappa_2^2)} \qquad \text{......(2),}$$

where A is a positive constant, κ_1, κ_2 are the free periods of the matter, and $\kappa_2 > \kappa_1$. When $\tau > \kappa_2$, V is real; but when τ lies between κ_1 and κ_2, V will be imaginary; and when $\tau < \kappa_1$, V will

be again real. Hence a medium, in which the velocity of light is represented by (2), would be transparent to waves of light, whose periods lie between ∞ and κ_2 and between κ_1 and $\dot{0}$, but would be opaque to waves whose periods lie between κ_2 and κ_1; and (2) might therefore represent a medium, which has an absorption band in the green.

Colours of Natural Bodies.

304. The colours of natural bodies are due to two distinct causes; first, because they absorb certain rays of the spectrum, secondly, because their structure is irregular. That the colours of natural bodies do not arise from their being unable to reflect light of the same colour as that which they absorb, can be shown by examining the surface of a solution of sulphate of copper, enclosed in a vessel whose sides are painted black; for it will be found that the solution appears to be black instead of blue, as would be the case if it reflected light of the same colour as that which it transmits. Moreover the image of a white object reflected at the surface, appears white instead of coloured. If however the solution be rendered turbid by the addition of a little powdered chalk, the liquid immediately appears to be of a brilliant blue. The addition of the chalk produces what is equivalent to an irregularity in the structure of the liquid, and its colour is thereby made manifest.

When white light is incident upon a substance having an irregular structure, a small portion of the light is reflected at the surface, but by far the greater portion will enter the substance. Of this latter portion, a part will pass between the particles, and another part will be refracted by them. The refracted portion will become coloured owing to the absorption of certain kinds of light, and will then be reflected in an irregular manner at the surfaces of the different particles. Hence the light, which taken as a whole, comes from the surface of the substance, exhibits the colour of the particular kind of light, which the substance does not absorb. The red colour of the poppy is due to the fact, that the leaves of its flowers contain a juice, which absorbs blue, green, and a portion of the yellow rays; and if this flower be held in the red end of the spectrum it appears to be of a brilliant red, whilst if it be held in the green or blue it appears to be black,

since under these circumstances the poppy absorbs nearly the whole of the light which falls upon it. The colour of white objects is due to the fact, that they absorb none of the rays; whilst the absence of colour of black objects, arises from their power of absorbing rays of every colour.

Dichromatism.

305. There are certain substances, which appear to be of one colour when white light is passed through a stratum of moderate thickness, but which appear to be of a different colour when the thickness is large. For example, light transmitted through glass coloured with cobalt, appears to be blue when the thickness is small, purple when the thickness is greater, and deep red when the thickness is large. If on the other hand, a solution of chlorophyll (that is, the green colouring matter of leaves), in alcohol be employed, the transmitted light appears to be of a bright emerald green when the thickness is small, and red when the thickness is large. This phenomenon is called *dichromatism.*

306. Dichromatism was first explained by Sir J. Herschel in the following manner. Cobalt glass is obviously opaque to yellow and green light; and although it is capable of transmitting blue and red light, it absorbs blue light to a greater extent than red. Let us consider a thin stratum of the substance, whose thickness is dx; let I, $I+dI$ be the intensities of the light which enters and emerges from the stratum, and let q represent the proportion of light of unit intensity, which is absorbed by a stratum of unit thickness. Then $qIdx$ is the proportion of light of intensity I, which is absorbed by a stratum of thickness dx; whence

$$dI = -qIdx;$$

and therefore

$$I = A\epsilon^{-qx},$$

where A is the intensity of the light incident upon the substance. This is the value of the intensity of the light after it has passed through a stratum of thickness x. The quantity q is called the coefficient of absorption of the substance, and is dependent upon the colour of the incident light.

307. Let us now denote the values of the quantities for blue and red light by the suffixes b and r. Then for cobalt glass, q may

be regarded as infinite for all colours but blue and red, but is finite for these colours, hence

$$\frac{I_b}{I_r} = \frac{A_b}{A_r} \epsilon^{-(q_b - q_r)x}.$$

Since the blue rays are entirely absorbed when the thickness is considerable, it follows that $q_b > q_r$. Since the glass appears to be blue when the thickness is small, it follows that $A_b > A_r$. This must be interpreted to mean, that out of the rays of different colours which fall upon the glass, a far greater number of blue rays are capable of being transmitted than of red. In other words, cobalt glass transmits a very large portion of blue extremity of the spectrum and very little of the red, but the coefficient of absorption of the blue rays is greater than that of the red.

308. In the case of chlorophyll, the coefficient of absorption is very large for all rays but green and red, and is greater for green than red; but this substance is capable of transmitting a larger portion of the green part of the spectrum, than of the red.

Anomalous Dispersion.

309. We have already pointed out, that the order of the colours in the solar spectrum is violet, indigo, blue, green, yellow, orange, red; and that the violet is the most refracted, and the red is the least. There are however certain substances, in which the order of the colours in the spectrum produced by refracting sunlight through them, is different from that produced by glass, and the majority of transparent media. The dispersion produced by such substances is called *anomalous dispersion.*

310. Anomalous dispersion appears to have been first observed by Fox Talbot[1] about 1840, but the discovery excited no attention. It was next observed by Leroux[2] in 1862, who found that vapour of iodine refracted red light more powerfully than violet. This substance absorbs all colours except red and violet, and it was observed that the order of the colours in the spectrum, beginning at the top, was red, then an absorption band, and then violet. The indices of refraction, as determined by Hurion[3], are

$$\mu_r = 1{\cdot}0205, \quad \mu_v = 1{\cdot}019.$$

[1] See *Proc. R. S. E.* 1870; and Tait, Art. Light, *Encycl. Brit.*

[2] *C. R.* 1862; and *Phil. Mag.* Sept. 1862.

[3] *Journ. de Phys.* 1st Series, Vol. VII. p. 181.

311. Anomalous dispersion is most strongly marked in solutions of the aniline dyes in alcohol. Christiansen[1] discovered in 1870, that it was produced by fuchsine, which is one of the rose aniline dyes; for when sunlight was passed through a prism containing a solution of this substance, it was found that the order of the colours was[2] indigo, green, red and yellow, the indigo being the least deviated. The amount of anomalous dispersion increases with the concentration of the solution, as is shown in the following table of the indices of refraction.

Fuchsine solution	B		D	F	G	H
18·8 per cent.	1·450	1·502	1·561	1·312	1·285	1·312
2·5 per cent.	1·384	—	1·419	1·373	1·367	1·373

From this table, we see that the line D is more refracted than any of the others, and that the violet is less refracted than the red.

312. Kundt[3] afterwards showed, that blue, violet and green aniline, indigo, indigo-carmine, cyanine, carmine, permanganate of potash, chlorophyll and a variety of other substances exhibited anomalous dispersion. The following table gives some of the indices of refraction found by him[4].

	A	B	C	D	E	F	G	H
Cyanine 1·22 per cent. solution	1·3666	1·3691	1·3714	—	1·3666	1·3713	1·3757	1·3793
Do. a stronger solution	1·3732	1·3781	1·3831	—	1·3658	1·3705	1·3779	1·3821
Fuchsine	1·3818	1·3873	1·3918	1·3982	—	—	1·3668	1·3759
Permanganate of Potash	1·3377	1·3397	1·3408	1·3442	—	—	1·3477	1·3521

[1] *Pogg. Ann.* Vol. CXLI. p. 479; and *Phil. Mag.* March, 1871, p. 244.

[2] *Ibid.* Vol. CXLIII. p. 250. See also Wiedermann, *Ber. der Sächs. Gesell. math.-phys. Cl.* Vol. I. 872; G. Lundquist, *Nova acta reg. Soc. Sc. Upsaliensis* [3] Vol. IX. Part II. (1874); *Jour. de Physique*, Vol. III. p. 352 (1874).

[3] *Pogg. Ann.* Vols. CXLII. p. 163; CXLIII. pp. 149, 259; CXLIV. p. 128; CXLV. p. 164.

[4] *Pogg. Ann.* Vol. CXLV. p. 67.

The preceding table gives a general idea of the condition of the spectrum, and shows that for cyanine the line E is the least refracted. In the case of cyanine and fuchsine, the lower end of the spectrum is blue, then comes an absorption band, and afterwards red and orange; so that the blue is least refracted, the green and some of the yellow are absorbed, and the orange is the most refracted. In the spectrum produced by permanganate of potash, there is a slight amount of anomalous dispersion between D and G; for Kundt found, that the indices of refraction for green and blue were 1·3452 and 1·3420 respectively, showing that the blue is less refracted, than the green in the neighbourhood of D. In the region between D and G there are also several absorption bands.

313. By means of his experiments, Kundt deduced the following law:—

On the lower or less refrangible side of an absorption band, the refractive index is abnormally increased; whilst on the upper or more refrangible side, it is abnormally diminished.

In order to clearly understand this law, let us revert to the spectrum produced by fuchsine. In this substance the absorption is very strong between D and F, that is in the green portion of the spectrum; and on looking at the table, we see that the red and orange rays, which lie below the green in the spectrum produced by a glass prism, lie above it in the case of fuchsine; whilst the violet rays lie below the green. The refrangibility of the red and orange rays is therefore abnormally increased, whilst that of the violet is abnormally diminished.

Selective Reflection.

314. We have already pointed out, that the colours of natural bodies arise from the fact that they *absorb* certain kinds of light; there is however another class of substances, which strongly reflect light of certain colours, whilst they very slightly reflect light of other colours. The phenomenon exhibited by these substances is called *selective reflection.*

315. Selective reflection appears to have been first discovered

by Haidinger[1]. It was subsequently studied by Stokes[2]; and the experiments of Kundt, which have already been referred to[3], show, that it is exhibited by most substances which produce anomalous dispersion. In fact absorption, anomalous dispersion and selective reflection are so closely connected together, that they must be regarded as different effects of the same cause, and consequently ought to be capable of being explained by the same theoretical considerations.

316. The properties of substances, which exhibit selective reflection, may be classified under the following three laws:

I. *Those rays which are most strongly reflected, when light is incident upon the substance, are most strongly absorbed, when light is transmitted through the substance.*

II. *When the incident light is plane polarized in any azimuth, the reflected light exhibits decided traces of elliptic polarization.*

III. *When sunlight is reflected, and the reflected light is viewed through a Nicol's prism, whose principal section is parallel to the plane of incidence, the colour of the reflected light is different from what it is, when viewed by the naked eye.*

Since the reflected light is elliptically polarized, it follows that selective reflection is accompanied by a change of phase of one or both the components of the incident light.

317. The properties of substances, which exhibit selective reflection, resemble those of metals, as will be explained in the Chapter on Metallic Reflection. For in the first place, metals strongly absorb light, and powerfully reflect it; and in the second place light reflected by a metallic surface is always elliptically polarized, unless the plane of polarization of the incident light is parallel or perpendicular to the plane of incidence. The optical properties of these substances appear to occupy a position, intermediate between ordinary transparent media and metals; and on this account, selective reflection is sometimes called *quasi-metallic reflection.*

[1] Ueber den Zusammenhang der Körperfarben, oder des farbig durchgelassenen, und der Oberflächenfarben, oder des zurückgeworfenen Lichtes gewisser Körper. *Proc. Math. and Phys. Class of the Acad. of Sciences at Vienna* 1852, and the papers there cited.

[2] *Phil. Mag.* (4) VI. p. 393.

[3] *Ante*, p. 297.

318. On the other hand, metallic reflection produces very little chromatic effect, whilst the peculiarities of substances which produce selective reflection principally consist in chromatic effects. Moreover reflection from ordinary transparent substances, is considerably weakened by bringing them into optical contact with another having nearly the same refractive index; but in the case of quasi-metallic substances, the colours which they reflect, are brought out more strongly by placing them in optical contact with glass or water.

319. That there are certain substances, which strongly reflect light of the same periods as those which they absorb, is strikingly exemplified in the case of permanganate of potash. Stokes found[1], that when the light transmitted by a weak solution is analysed by a prism, there are five absorption bands, which are nearly equidistant, and lie between D and F. The first band, which lies a little above D, is less conspicuous than the second and third, which are the strongest of the set. If however light incident at the polarizing angle is reflected from permanganate of potash, and is then passed through a Nicol, placed so as to extinguish the light polarized in the plane of incidence, the residual light is green; and when it is analysed by a prism, it *shows bright bands where the absorption spectrum shows dark ones.*

320. Safflower-red or carthamine is another example of a substance which exhibits selective reflection. Stokes found, that this substance powerfully absorbed green light, but reflected a yellowish green light; and that when red light polarized at an azimuth of 45° was incident upon this substance, the reflected light was sensibly plane polarized, but when green or blue light, polarized in the same azimuth, was substituted, the reflected light was elliptically polarized. It further appeared, that the chromatic effects of this substance were different, according as the incident light was polarized in or perpendicularly to the plane of incidence; for when the incident light was polarized perpendicularly to the plane of incidence, the reflected light was of a very rich green colour, but when it was unpolarized the reflected light was yellowish-green.

Similar results were obtained by using a compound of iodine and quinine called herapathite, which was discovered by Dr Herapath of Bristol, and which strongly absorbs green light.

[1] *Phil. Mag.* Vol. VI. (18) p. 293.

321. The effect of bringing a transparent medium into optical contact with a quasi-metallic substance, may be illustrated by depositing a little safflower-red upon a glass plate, and allowing it to dry; when it will be found that the surface of the film which is in contact with air, is of a yellowish-green colour; whilst the surface in contact with glass, reflects light of a very fine green inclining to blue. Similar effects are produced with herapathite and platino-cyanide of magnesium. The latter crystal is one of a class of special optical interest, since it is doubly refracting, doubly absorbing, doubly metallic and doubly fluorescent.

322. Further experiments upon selective reflection have been made by Kundt[1], who found that it was strongly exhibited by the aniline dyes and other substances, which produce anomalous dispersion. The following table[2] shows the colour of the transmitted and reflected light, when the latter is viewed with the naked eye and through a Nicol's prism, adjusted so as to extinguish the component polarized in the plane of incidence.

Substance.	Transmitted.	Reflected.	Reflected & passed through a Nicol.
Rose aniline or fuchsine	Rose	Green	Peacock blue
Mauve aniline	Mauve	Apple green	Emerald green
Malachite green	Deep green	Plum colour	Orange gold
Blue aniline .	Blue	Bronze	Olive green

Fluorescence.

323. When common light is incident upon a solution of sulphate of quinine in water, it is found that the surface of the liquid exhibits a pale blue colour, which extends a short distance into the liquid; if however the light which is refracted by the substance, and has therefore passed through the thin coloured stratum, is allowed to fall upon the surface of a second solution of sulphate of quinine, the effect is no longer produced.

The peculiar action which sulphate of quinine, as well as

[1] *Ante*, p. 297, footnote.
[2] Glazebrook's *Physical Optics*, p. 273.

certain other substances, produces upon light, is called *fluorescence.* It was first discovered by Sir David Brewster[1] in 1833, who observed that it was produced by chlorophyll, and also by fluor spar. Sir J. Herschel[2] found that fluorescence was produced by quinine, but the subject was not fully investigated, until it was taken up by Sir G. Stokes[3] in 1852.

324. To examine the nature of fluorescence produced by quinine, Stokes formed a spectrum by means of a slit and a prism, and filled a test tube with the solution, and placed it a little beyond the red extremity of the spectrum. The test tube was then gradually moved up the spectrum, and no traces of fluorescence were observed, as long as the tube remained in the more luminous portion; but on arriving at the violet extremity, a ghost-like gleam of pale blue light shot right across the tube. On continuing to move the tube, the light at first increased in intensity, and afterwards died away, *but not until the tube had been moved a considerable distance into the invisible ultra-violet rays.* When the blue gleam of light first made its appearance, it extended right across the tube, but just before disappearing, it was observed to be confined to an excessively thin stratum, adjacent to the surface at which the light entered.

325. This experiment shows that in the case of quinine, fluorescence is produced by violet and ultra-violet light, and also that it is due to a change in the refrangibility of the incident light. Stokes also found, that quinine was exceedingly opaque to those rays of the spectrum which lie above the line H, that is to those rays by which fluorescence is produced. This explains why light, which has been passed through a solution of quinine, is incapable of producing fluorescence, for the solution absorbs the rays which give rise to this phenomenon.

326. The effect may accordingly be summarized as follows. Quinine is transparent to the rays constituting the lower or more luminous portion of the spectrum, but it strongly absorbs the ultra-violet rays, and gives them out again as rays of lower refrangibility. The latter circumstance enables the eye to take cognizance of the invisible ultra-violet rays; for if this portion of

[1] *Trans. R. S. E.* Vol. XII. p. 542.
[2] *Phil. Trans.* 1845.
[3] *Ibid.* 1852, p. 463.

the spectrum is passed through a fluorescent substance, it is converted into luminous rays, which are visible, and can be examined by the eye. By this method Stokes was able to make a map of the fixed lines in the ultra-violet region.

327. Fluorescence is also produced by a number of other substances, among which may be mentioned decoction of the bark of the horse-chestnut, green fluor spar, solution of guaiacum in alcohol, tincture of turmeric, chlorophyll, yellow glass coloured with oxide of uranium &c. It must not however be supposed, that the light produced by fluorescence is of the same colour for all substances, since as a matter of fact, it varies for different substances. Thus the fluorescence produced by chlorophyll consists of red light, showing that this substance converts green and blue light into red light.

328. As the result of his experiments, Stokes was led to the following law, viz.;—*When the refrangibility of light is changed by fluorescence, it is always lowered and never raised.*

Whether this law is absolutely general has lately been doubted; and there appears to be some evidence, that exceptions to it exist.

329. We have already called attention to the fact, that the phenomena of dispersion, absorption and the like, are caused by the molecules of matter being set in motion by the vibrations of the ether. Now if the molecular forces depended upon the first power of the displacements, it would follow from Herschel's theorem, that the period of the forced vibrations would be equal to that of the force; if however the molecular forces depended upon the squares or higher powers of the displacements, Herschel's theorem would be no longer true, and under these circumstances Stokes suggested, that fluorescence arises from the fact, that the forces are such, that powers of the displacements higher than the first cannot be neglected. We have already pointed out, that ultra-violet light produces strong chemical effects. Now the molecules of most organic substances consist of a number of chemical atoms connected together, and forming a system of more or less complexity, which is stable for some disturbances but unstable for others. For instance, an ordinary photographic plate is fairly stable for disturbances produced by sodium light, but unstable for those produced by violet light. It is therefore not

unreasonable to suppose, that the amplitudes of the vibrations communicated by ultra-violet light to the molecules, and to the atoms composing them, of a substance like quinine, should be of such far greater magnitude, than those communicated by light of less refrangibility, that the molecular forces produced under the former circumstances cannot be properly represented by forces proportional to the displacements. If this be the case, the period of the forced vibrations will no longer be equal to that of the force.

330. When a molecule is set into vibration by ethereal waves, the vibrations of the molecule will give rise to secondary waves in the ether. The periods of these secondary waves must necessarily be the same as those of the molecules by which they are produced; for Herschel's theorem applies to vibrations communicated to the ether, although it does not necessarily apply to vibrations communicated to the molecules. And if the periods of the secondary waves are longer than those of the waves impinging on the molecules, these waves will be capable of producing the sensation of light, provided their periods lie within the limits of sight, even though the periods of the impinging waves are too short to be visible. We can thus obtain a mechanical explanation of the way in which fluorescence is produced, but at the same time the following illustration will make the matter clearer.

331. The equation of motion of a molecule, which is under the action of molecular forces, which are proportional to the cube of the displacement, and which is also under the action of a periodic force, is

$$\frac{d^2y}{dt^2}+\mu^2y^3=F\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

The particular solution gives the forced vibration, whilst the complementary function gives the free vibration. The determination of the particular solution when $F=A\epsilon^{\iota pt}$, where A is an arbitrary constant, would be difficult; but as the above equation is given as an illustration, and not for the purpose of constructing a theory, we shall suppose that the force is represented by

$$\frac{2p^3}{3\mu\sqrt{3}}\cos 3pt.$$

The particular solution will then be found to be

$$y=\frac{2p}{\mu\sqrt{3}}\cos pt.$$

From this result we see, that the period of the forced vibration is three times that of the force; accordingly the secondary waves will be of longer period, and consequently less refrangible, than the impinging waves.

Calorescence.

332. This phenomenon is the reverse of fluorescence, and consists in the conversion of waves of long period into waves of shorter period. Calorescence is well exhibited by the experiment of Tyndall already described under the head of spectrum analysis[1], in which the light from an electric lamp is sifted of the luminous rays, by passing it through a solution of iodine in disulphide of carbon, which only allows the infra-red rays to pass through.

333. In order to obtain a mechanical model which will illustrate calorescence, we may revert to the differential equation (1). It can be verified by trial, that the complementary function is

$$y = a \operatorname{cn}(a\mu t + \alpha), \quad k = 2^{-\frac{1}{2}} \qquad \text{.....................(2)},$$

where a and α are the constants of integration. From this result it follows, that the amplitude of the free vibration is a, and its period is $4K/\mu a$, which is inversely proportional to the amplitude. Hence the period diminishes as the amplitude increases.

Equation (2) may still be regarded as the complete solution of (1), provided we suppose that a and α, instead of being constants, are functions of the time, and their values might be found by the method of variation of parameters. If now, we suppose that the molecular forces are such, that a increases slowly with the time, we may illustrate the conversion of waves of dark heat into waves of light. When the waves of dark heat first fall on the substance and the forces begin to act, the amplitude is very small, and consequently the period τ is very large. On both these grounds therefore, the vibrations are incapable of affecting the senses. As the forces continue to act, the amplitude increases, whilst the period diminishes, and the vibrations become sensible as heat; in other words the substance begins to get hot. As this process continues, the substance becomes red hot, and then intensely luminous. As the amplitudes cannot go on increasing indefinitely with the time, we must suppose that after the expiration of a certain period, the

[1] Ante, p. 286.

condition of the substance changes owing to liquefaction or vapourization, and that the equation by means of which the original state of things was represented, no longer holds good.

Phosphorescence.

334. When light is incident upon certain substances, such as the compounds of sulphur with barium, calcium or strontium, it is found that they continue to shine, after the light has been removed. This phenomena is called *phosphorescence.*

Phosphorescence is closely allied to fluorescence, inasmuch as it is usually produced by rays of high refrangibility, and the refrangibility of the phosphorescent light is generally less than that of the light by which it is produced. The principal distinction between the two phenomena is, that fluorescence lasts only as long as the exciting cause continues, whilst phosphorescence lasts some time after it has been removed.

335. In order to give a mechanical explanation of phosphorescence, we shall employ an acoustical analogue, which will frequently be made use of, and which will be fully worked out in § 337. Let plane waves of sound be incident upon a sphere, whose radius is small in comparison with the lengths of the waves of sound; and let the sphere be attached to a spring, so as to be capable of vibrating parallel to the direction of propagation of the waves. Then it is known[1], and will hereafter be proved, that the effect of the waves of sound will be to cause the sphere to vibrate. If the strength of the spring is such, that the force due to it is proportional to the displacement of the sphere, the forced period of the latter will be equal to that of the impinging waves of sound; if the law of force depends upon some power of the displacement, the forced period of the sphere will be different; but in either case secondary waves will be thrown off. These secondary waves will travel away into space carrying energy with them, which has been in the first instance communicated to the sphere by the incident waves, and then communicated back again to the air in the form of secondary waves. If the cause which produces the incident waves be removed, the sphere will still continue to vibrate, but it

[1] Lord Rayleigh, *Theory of Sound*, Ch. XVII.

cannot go on vibrating indefinitely, because the energy which it possessed at the instant at which the incident waves were stopped, will gradually be used up in generating secondary waves, and will be carried away into space by them; hence the sphere will ultimately come to rest, and no more secondary waves will be produced.

Now although the molecules of a *non-phosphorescent* substance cannot be supposed to come to rest immediately the exciting cause is removed, yet the time during which they continue to be in motion is too short to be observed; but owing to the peculiar molecular structure of *phosphorescent* substances, the molecules remain in motion for a longer period. Hence a luminous glimmer exists for some time after the incident light has been cut off.

CHAPTER XVII.

THEORIES BASED ON THE MUTUAL REACTION BETWEEN ETHER AND MATTER.

336. IN the present Chapter, we shall give an account of some of the attempts which have been made to explain on dynamical grounds the phenomena described in the previous Chapter.

It may be regarded as an axiom, that when ethereal waves impinge upon a material substance, the molecules of the matter of which the substance is composed, are thrown into a state of vibration. This proposition is quite independent of any hypothesis, which may be made respecting the constitution of the ether, the molecular forces called into action by the displacements of the molecules of matter, or the forces arising from the action of ether upon matter. It may therefore be employed as the basis of a theory, in which the ether is regarded, either as a medium possessing the properties of an elastic solid, or as one which is capable of propagating electromagnetic disturbances as well as luminous waves. The difficulties of constructing theories of this description arise, not only from the fact that the properties of the ether are a question of speculation, but also because the forces due to the action of matter upon matter, and of ether upon matter, are unknown.

During the last five and twenty years, numerous attempts have been made by continental writers to develop theories of this description, and an account of them will be found in Glazebrook's *Report on Optical Theories*[1]. It cannot, I think, be said that any of these theories are entirely satisfactory; but at the same time,

[1] *Brit. Assoc. Rep.* 1886.

they clearly indicate the direction, in which we must look for an explanation of the phenomena, which they attempt to account for.

337. As an introduction to the subject, we shall work out and discuss the problem of the sphere vibrating under the action of plane waves of sound, which has been referred to in the last Chapter. The problem itself was first solved by Lord Rayleigh[1], and has been reproduced by myself[2] in an approximate form; but there are several additional points which require consideration.

Let c be the radius of the sphere; let $\kappa = 2\pi/\lambda$, where λ is the wave-length; and let a be the velocity of sound. Then the velocity potential of the incident waves may be taken to be $\phi'\epsilon^{\iota\kappa at}$, where $\phi' = \epsilon^{\iota\kappa x}$. Now if $\mu = \cos\theta$,

$$\epsilon^{\iota\kappa x} = \Sigma_0^\infty F_n(c) P_n(\mu),$$

where P_n is the zonal harmonic of degree n, and[3]

$$F_n(c) = \frac{(\iota\kappa c)^n}{1\,.\,3\ldots(2n-1)}\left\{1 - \frac{\kappa^2c^2}{2\,.\,2n+3} + \frac{\kappa^4c^4}{2\,.\,4\,.\,2n+3\,.\,2n+5} - \ldots\right\} \quad (1).$$

If $\phi\epsilon^{\iota\kappa at}$ be the velocity potential of the secondary waves, we may assume[4]

$$\phi = \Sigma A_n \psi_n P_n,$$

where

$$\psi_n = \frac{\epsilon^{-\iota\kappa r}}{r} f_n(\iota\kappa r),$$

and f_n is the function defined by (40) of § 231.

If X be the resistance due to the pressure of the air,

$$\begin{aligned} X &= -2\pi\rho c^2 \int_0^\pi (\phi' + \phi)\, \iota\kappa a \epsilon^{\iota\kappa at} \cos\theta \sin\theta d\theta \\ &= -2\iota\pi\rho\kappa a c^2 \epsilon^{\iota\kappa at} \int_{-1}^1 (\phi' + \phi)\, \mu d\mu \\ &= -M'\iota\kappa a c^{-1} \{F_1(c) + A_1\psi_1(c)\}\, \epsilon^{\iota\kappa at} \ldots\ldots\ldots\ldots (2), \end{aligned}$$

where M' is the mass of the fluid displaced.

If $V\epsilon^{\iota\kappa at}$ be the velocity of the sphere, the boundary condition gives

$$A_1 \frac{d\psi_1}{dc} + \frac{dF_1}{dc} = V.$$

[1] *Proc. Lond. Math. Soc.* vol. IV. p. 253.

[2] *Elementary Hydrodynamics and Sound*, § 167.

[3] Lord Rayleigh, *Theory of Sound*, Ch. XVII.

[4] Stokes, On the communication of vibrations from a vibrating body to the atmosphere. *Phil. Trans.* 1868.

Now $$\psi_1 = \frac{\epsilon^{-\iota\kappa c}}{c}\left(1 + \frac{1}{\iota\kappa c}\right),$$

$$\frac{d\psi_1}{dc} = \frac{\iota\kappa\epsilon^{-\iota\kappa c}}{c}\left(\frac{2}{\kappa^2c^2} - \frac{2}{\iota\kappa c} - 1\right),$$

whence $$A_1\psi_1 = -\frac{(V - dF_1/dc)(1 + \iota\kappa c)c}{2 + 2\iota\kappa c - \kappa^2c^2}$$

$$= -\frac{(V - dF_1/dc)(2 + \kappa^2c^2 - \iota\kappa^3c^3)c}{4 + \kappa^4c^4}.$$

We shall now suppose, that the radius of the sphere is so small in comparison with the wave-length, that the fourth and higher powers of κc may be neglected. Hence if ξ be the displacement of the sphere,

$$\iota\kappa ac^{-1}M'A_1\psi_1\epsilon^{\iota\kappa at} = -\tfrac{1}{4}M'\left[\{(2 + \kappa^2c^2)\ddot{\xi} + \kappa^4c^3a\dot{\xi}\} - \{\iota\kappa a(2 + \kappa^2c^2) + \kappa^4c^3a\}\epsilon^{\iota\kappa at}\,dF_1/dc\right]\ldots\ldots\ldots(3).$$

Substituting the value of F_1 from (1), and remembering that $\kappa a = 2\pi/\tau$, where τ is the period of the waves of sound, we finally obtain from (2) and (3),

$$X = \tfrac{1}{4}M'\left\{(2 + \kappa^2c^2)\ddot{\xi} + \frac{2\pi}{\tau}\kappa^3c^3\dot{\xi}\right\} + \frac{4\pi^2M'}{a\tau^2}(\tfrac{3}{2} - \tfrac{1}{4}\iota\kappa^3c^3)\epsilon^{2\iota\pi t/\tau}.$$

If the force due to the spring is proportional to the displacement of the sphere, it follows that this force is equal to $4\pi^2M\xi/\tau'^2$, where τ' is the free period of the sphere; whence the equation of motion is

$$\{M + \tfrac{1}{4}M'(2 + \kappa^2c^2)\}\ddot{\xi} + \frac{M'\pi}{2\tau}\kappa^3c^3\dot{\xi} + \frac{4\pi^2M'}{a\tau^2}(\tfrac{3}{2} - \tfrac{1}{4}\iota\kappa^3c^3)\epsilon^{2\iota\pi t/\tau} + \frac{4\pi^2M}{\tau'^2}\xi = 0\ldots\ldots(4).$$

To integrate this equation, assume $\xi = A\epsilon^{2\iota\pi t/\tau}$, then

$$A = \frac{M'\tau'^2(\tfrac{3}{2} - \tfrac{1}{4}\iota\kappa^3c^3)/a}{M(\tau'^2 - \tau^2) + M'(\tfrac{1}{2} + \tfrac{1}{4}\kappa^2c^2)\tau'^2 - \tfrac{1}{4}\iota\kappa^3c^3M'\tau'^2}\ldots\ldots(5).$$

338. From these results we draw the following conclusions.

Equation (4), which is the equation of motion, contains a viscous term, that is a term proportional to the velocity. This term arises from the circumstance, that the sphere is continually losing the energy which it receives from incident waves, by generating secondary waves, which travel away into space carrying energy with them. If therefore the supply of energy be stopped,

by removing the cause which produces the impinging waves, the sphere will gradually get rid of all its energy, and will ultimately come to rest. The time which elapses before the sphere comes to rest, will depend upon the value of the modulus of decay; if this quantity is small, the vibrations will die away almost instantaneously; but if the modulus of decay is larger, the vibrations will continue for a sufficient time to enable our senses to take cognizance of them.

339. Now whatever supposition we make concerning the mutual reaction between ether and matter, it is practically certain that the motion of a molecule of matter will be represented by an equation, whose leading features are the same as (4), although the equation itself may be of far greater complexity. We therefore infer, that the molecular structure of non-phosphorescent substances is such, that the modulus of decay is so small as to be inappreciable; whilst the molecular structure of phosphorescent substances is such, that the modulus of decay is considerably larger.

340. We must now consider the amplitude of the vibrations of the sphere, which is given by (5). The density of all gases is exceedingly small, compared with the densities of substances in the solid or liquid state; consequently M' is very small compared with M. Hence the amplitude of the sphere is exceedingly small in comparison with that of the incident waves (which has been taken as unity), unless τ and τ' are nearly equal. When $\tau = \tau'$, the large term in the denominator disappears, and A is approximately equal to $3/a$. Under these circumstances, the amount of energy communicated by the incident waves to the sphere, is very much greater than what it would have been, if the difference between τ and τ' were considerable.

341. Let us now consider a medium, such as a stratum of sodium vapour. We may conceive the molecules of the medium to be represented by a very large number of small spheres, and the molecular forces to be represented by springs. The medium will therefore have one or more free periods of vibration. The interstices between the molecules are filled with ether, which is represented by the atmosphere. When waves of light pass through the medium, the molecules will be set into vibration, and a certain amount of energy will be absorbed by them; but since the mass of

a molecule is exceedingly large compared with the mass of the ether which it displaces, the amplitudes of the vibrations of the molecules will be very small, and very little energy will be absorbed, unless the period of the waves is equal, or nearly so, to one of the free periods of the system. But in the case of equality of the free and forced periods, the amplitudes will be so large, that a great deal of energy will be taken up by the molecules, and of the energy which entered the stratum of vapour, very little will emerge. Light will therefore be absorbed. The absorption bands produced by sodium vapour may therefore be explained, by supposing that sodium vapour has two free periods, which are very nearly equal to one another, and accordingly produce the double line D in its absorption spectrum. Hydrogen, on the other hand, has three principal free periods, which are separated from one another by considerable intervals.

The occurrence of an imaginary term in the denominator of A shows, that A can never become infinite for any real value of τ. This remark will be found of importance later on.

Lord Kelvin's Molecular Theory.

342. We shall now consider a theory, which was developed by Sir W. Thomson, now Lord Kelvin, in his lectures on Molecular Dynamics, delivered at Baltimore in 1884.

The molecules of matter are represented by a number of hollow spherical shells, connected together by zig-zag massless springs; and the outermost shell is connected by springs to a massless spherical envelop, which is rigidly connected with the ether. The space between any two shells is supposed to be a vacuum, and transparent and other substances are supposed to consist of a great number of such shells, which may be imagined to represent the molecules.

The degree of complexity of the molecule will depend upon the number of shells which it contains; and we can by this means represent chemical compounds of every degree of complexity.

We shall first of all investigate the motion of a single molecule, on the supposition that the centres of all the spherical shells are vibrating along a fixed straight line. We shall also suppose, that the force exerted by the springs joining two consecutive shells is proportional to their relative displacements; and that the force

exerted by the ether on the envelop is proportional to $\xi - x_1$; where x_1 is the displacement of the outermost shell, and ξ may be regarded indifferently, either as the displacement of the envelop, or of the ether in contact with it.

343. Let $m_i/4\pi^2$ be the mass of the i^{th} shell, x_i its displacement, C_i the strength of the spring connecting the i^{th} and $(i-1)^{\text{th}}$ shells. Then the equations of motion of the system of shells will be

$$\frac{m_1}{4\pi^2}\ddot{x}_1 = C_1(\xi - x_1) - C_2(x_1 - x_2)$$

$$\frac{m_2}{4\pi^2}\ddot{x}_2 = C_2(x_1 - x_2) - C_3(x_2 - x_3) \qquad (6).$$

$$\frac{m_i}{4\pi^2}\ddot{x}_i = C_i(x_{i-1} - x_i) - C_{i+1}(x_i - x_{i+1})$$

If τ be the period, and if we put

$$a_i = m_i/\tau^2 - C_i - C_{i+1} \qquad (7),$$

the equations of motion will reduce to the form

$$-C_i x_{i-1} = a_i x_i + C_{i+1}x_{i+1} \qquad (8).$$

The first equation of the form (8) is

$$-C_1\xi = a_1 x_1 + C_2 x_2 \qquad (9),$$

so that we may regard x_0, as equal to ξ the displacement of the ether.

If we suppose that the j^{th} shell is attached to a fixed point, the j^{th} equation of motion will be

$$\frac{m_j}{4\pi^2}\ddot{x}_j = C_j(x_{j-1} - x_j) - C_{j+1}x_j,$$

whence
$$-C_j x_{j-1} = a_j x_j \qquad (10).$$

Although it is scarcely admissible to suppose, that the j^{th} shell is attached to a fixed point, yet if we suppose that the $(j+1)^{\text{th}}$ shell consists of a solid nucleus, whose mass is large compared with that of the other shells, its motion will be sufficiently small to be neglected.

There are $j-2$ equations of the form (8), which are obtained by putting $i = 2, 3, \ldots j-1$; and these equations together with (9) and (10) furnish altogether j equations, from which the $j-1$ quantities $x_2, x_3 \ldots x_j$ can be eliminated, and we shall thus obtain a relation between x_1 and ξ.

344. To perform the elimination, let

$$u_i = -\frac{C_i x_{i-1}}{x_i},$$

or

$$\frac{x_i}{x_{i-1}} = -\frac{C_i}{u_i} \text{..........................} (11).$$

Then (9), (8) and (10) become

$$-\frac{C_1 \xi}{x_1} = u_1 = a_1 - \frac{C_2^2}{u_2} \text{....................} (12),$$

$$u_i = a_i - \frac{C^2_{i+1}}{u_{i+1}} \text{......................} (13).$$

Also since $x_{j+1} = 0$; $u_{j+1} = \infty$, and therefore

$$u_j = a_j \text{..............................} (14).$$

From these equations we see, that u_1 can be expressed in the form of the continued fraction

$$-\frac{C_1 \xi}{x_1} = a_1 - \frac{C_2^2|}{a_2} - \frac{C_3^2|}{a_3} - \text{......} \frac{C^2_{j-1}|}{a_{j-1}} - \frac{C^2_j}{a_j} \text{......} (15).$$

Putting for a moment δ for $d/d\tau^{-2}$, it follows from (7) that

$$\delta a_i = m_i;$$

and therefore by (13) and (14),

$$\delta u_i = m_i + \frac{C^2_{i+1}}{u^2_{i+1}} \delta u_{i+1},$$

$$\delta u_j = \delta a_j = m_j,$$

whence

$$\delta u_i = m_i + m_{i+1}\left(\frac{C_{i+1}}{u_{i+1}}\right)^2 + m_{i+2}\left(\frac{C_{i+1}C_{i+2}}{u_{i+1}u_{i+2}}\right)^2$$
$$+ \dots m_j \left(\frac{C_{i+1}C_{i+2}\dots C_j}{u_{i+1}u_{i+2}\dots u_j}\right)^2 \text{......} (16).$$

But from 11)

$$\frac{C_{i+1}}{u_{i+1}} = -\frac{x_{i+1}}{x_i},$$

$$\frac{C_{i+1}C_{i+2}}{u_{i+1}u_{i+2}} = \frac{x_{i+2}}{x_i},$$

also

$$\delta u_i = -\tfrac{1}{2}\tau^3 \frac{du_i}{d\tau},$$

whence

$$\frac{du_i}{d\tau} = -\frac{2}{\tau^3 x^2_i}(m_i x^2_i + m_{i+1} x^2_{i+1} + \dots m_j x^2_j) \text{......} (17).$$

From this equation we see, that $du_i/d\tau$ is always negative and therefore u_i diminishes as τ increases.

345. When τ is sufficiently small, all the u's are exceedingly large positive quantities; for since $u_{j+1} = \infty$, it follows from (7) and (14) that

$$u_j = \frac{m_j}{\tau^2} - C_j - C_{j+1},$$

so that u_j can always be made as large a positive quantity as we please, by taking τ small enough; whence it follows from (7) and (13), that all the u's can be made positive, provided τ is small enough. But if u_i is positive, we see from (11), that the signs of x_i and x_{i-1} must be different; accordingly when τ is very small, each shell is moving in the opposite direction to the two adjacent ones; also when u_i is large, the numerical value of x_{i-1} must be very much greater than that of x_i.

These considerations show, that when the period is exceedingly small, the vibrations of each shell, and also those of the outer massless envelop, which is supposed to be rigidly connected with the ether, are executed in opposite directions; and that the amplitudes of the vibrations of successive shells diminish with great rapidity, as we proceed inwards into the molecule.

It follows from (7) and (13) that as τ increases, a_i diminishes, whilst C^2_{i+1}/u_{i+1} increases; accordingly when τ has sufficiently increased, u_i will be zero. Now when u_i is zero, $u_{i-1} = -\infty$; and will therefore have passed through zero, and have changed sign for some value of τ, less than that for which u_i became zero. It therefore follows, that as τ increases from a value for which all the u's are positive, u_1 will be the first quantity which vanishes and changes sign, and that u_2 will be the next and so on.

346. In the problem we are considering, the motion is supposed to be produced by means of a forced vibration of amplitude ξ, and therefore when $x_1 = \infty$, $u_1 = 0$; but u_1 will also vanish when $\xi = 0$, hence the critical period for which u_1 vanishes is the *least* period of the *free* vibrations of the system, when the massless envelop is motionless. As soon as τ exceeds the first critical period, u_1 will become negative; and consequently the first shell and the massless envelop, will be moving in the same directions, whilst all the other shells will be moving in opposite directions. If now τ be supposed to still further increase, u_2 will diminish and finally vanish, in which case $u_1 = -\infty$, $x_1 = 0$. This is the second critical case; and the period of vibration is equal to the period of the free vibrations of the system when m_1 is fixed, and all the

other shells are vibrating in opposite directions; and this period is the least period of the possible free vibrations of the system under these conditions. The remaining critical cases can be discussed in a similar manner.

347. From (13) and (14), we have

$$\frac{1}{u_{j-1}} = \frac{a_j}{a_{j-1}a_j - C^2_j}.$$

From this equation we see, that $1/u_{j-1}$ is a fraction, whose numerator is a linear function, and whose denominator is a quadratic function of τ^{-2}. It therefore follows that $1/u_1$, or $-x_1/C_1\xi$, is a fraction whose numerator is a $(j-1)^{\text{th}}$, and whose denominator is a j^{th} function of τ^{-2}. Since u_1 is zero, when τ is equal to any one of the j periods of the free vibrations of the system, when the envelop is held fixed, it follows that the denominator of $1/u_1$ is expressible as the product of factors of the form $\kappa^2_i - \tau^2$, where $\kappa_1, \kappa_2 \ldots \kappa_j$ are the above mentioned free periods.

The value of $1/u_1$ may therefore be resolved into partial fractions, and may accordingly be expressed in the form

$$\frac{1}{u_1} = -\frac{x_1}{C_1\xi} = \frac{q_1}{\kappa_1^2/\tau^2 - 1} + \frac{q_2}{\kappa_2^2/\tau^2 - 1} + \ldots\ldots\ldots\ldots \quad (18),$$

where $q_1, q_2 \ldots$ are constants.

Writing for a moment D_i for $\kappa^2_i/\tau^2 - 1$, (18) becomes

$$u_1 = \left(\frac{q_1}{D_1} + \frac{q_2}{D_2} + \ldots\right)^{-1},$$

whence
$$-\frac{du_1}{d\tau^{-2}} = \frac{q_1\kappa_1^2/D_1^2 + q_2\kappa_2^2/D_2^2 + \ldots\ldots}{(q_1/D_1 + q_2/D_2 + \ldots\ldots)^2} \ldots\ldots\ldots \quad (19).$$

Now x_i is the amplitude of the i^{th} shell; if therefore we denote the actual displacement by x'_i, we shall have

$$x'_i = x_i \sin 2\pi t/\tau,$$

provided t be measured from the epoch at which each shell passes through its mean position. The energy in this particular configuration will be wholly kinetic; whence remembering that the mass of each shell is equal to $m_i/4\pi^2$, it follows that if E be the total energy,

$$\begin{aligned} E &= \frac{1}{8\pi^2}(m_1\dot{x}'^2_1 + m_2\dot{x}'^2_2 + \ldots\ldots) \\ &= \frac{1}{2\tau^2}(m_1x_1^2 + m_2x_2^2 + \ldots\ldots) \\ &= -\frac{x_1^2}{2\tau^2}\frac{du_1}{d\tau^{-2}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (20), \end{aligned}$$

by (17). Let

$$R^{-1} = 2E\tau^2/m_1 x_1^2,$$

so that R^{-1} denotes the ratio of the whole energy of the molecule to that of the first shell; then (19) becomes

$$\frac{m_1}{R} = \frac{q_1\kappa_1^2/D_1^2 + q_2\kappa_2^2/D_2^2 + \ldots\ldots}{(q_1/D_1 + q_2/D_2 + \ldots)^2}.$$

Hence if R_i denote the value of R when $\tau = \kappa_i$, we obtain

$$q_i = R_i\kappa^2{}_i/m_1$$

and (18) becomes

$$-\frac{x_1}{C_1\xi} = \frac{\tau^2}{m_1}\left(\frac{R_1\kappa_1^2}{\kappa_1^2 - \tau^2} + \frac{R_2\kappa_2^2}{\kappa_2^2 - \tau^2} + \ldots\right) \ldots\ldots\ldots (21).$$

348. Let us now imagine a medium, whose structure is represented by a very large number of molecules of the kind we have been considering; and let us suppose, that the interstices between the molecules are filled with ether, which is assumed to be a medium, whose motion is governed by the same equations as those of an elastic solid. Then if we confine our attention to a small element of the medium, which contains molecules and ether surrounding them, and for simplicity consider the propagation of waves parallel to the axis of x, the equation of motion of a particle of ether will be

$$\rho\frac{d^2w'}{dt^2} = n\frac{d^2w'}{dx^2}.$$

Integrating this equation throughout the volume of the element, we obtain

$$\rho\iiint \ddot{w}'dx\,dy\,dz = n\left[\iint\frac{dw'}{dx}\,dy\,dz\right]_1 - n\left[\iint\frac{dw'}{dx}\,dy\,dz\right]_2 \ldots (22).$$

The first surface integral on the right-hand side is to be taken over the outer boundary of the element, whilst the second is to be taken over the boundaries of each of the molecules. Let w be the mean value of w' within the element; then the values of w' at the points $x + \frac{1}{2}\delta x$ and $x - \frac{1}{2}\delta x$ will be

$$w + \tfrac{1}{2}\frac{dw}{dx}\delta x \text{ and } w - \tfrac{1}{2}\frac{dw}{dx}\delta x,$$

so that the first integral reduces to

$$n\frac{d^2w}{dx^2}\delta x\,\delta y\,\delta z.$$

The second integral represents the resultant of the forces, which each molecule exerts upon the ether; and if we represent the force due to a single molecule by $4\pi^2 C(\zeta' - w')\,dx'dy'dz'$, it follows that we may represent the resultant force due to all the molecules within the element by $4\pi^2 C(\zeta - w)\,\delta x\,\delta y\,\delta z$, where ζ is the mean value of the displacements of all the molecules. The equation of motion therefore becomes

$$\rho\frac{d^2w}{dt^2} = n\frac{d^2w}{dx^2} + 4\pi^2 C(\zeta - w) \quad\ldots\ldots\ldots\ldots\ldots (23).$$

349. To solve (23) assume

$$w = \xi\epsilon^{2\iota\pi/\tau\,.\,(x/V - t)}$$

$$\zeta = x_1\epsilon^{2\iota\pi/\tau\,.\,(x/V - t)}.$$

Substituting in (23), we obtain

$$\frac{n}{\rho V^2} = 1 + \frac{C\tau^2}{\rho}\left(\frac{x_1}{\xi} - 1\right).$$

Now $(n/\rho)^{\frac{1}{2}}$ is the velocity of light in vacuo, whence the left-hand side is equal to μ^2, where μ is the index of refraction. Hence if we substitute the value of x_1/ξ from (21), and write q_1 for $C_1R_1\kappa_1^2/m_1$, we shall finally obtain

$$\mu^2 = 1 + \frac{C\tau^2}{\rho}\left(\frac{q_1\tau^2}{\tau^2 - \kappa_1^2} + \frac{q_2\tau^2}{\tau^2 - \kappa_2^2} + \ldots\ldots - 1\right) \ldots\ldots (24).$$

350. This equation determines the index of refraction in terms of the period. To apply it to ordinary dispersion, we shall write it in the form

$$\mu^2 = 1 + \frac{C}{\rho}\left\{q_1\kappa_1^2 - (1 - q_1)\tau^2 + \frac{q_1\kappa_1^4}{\tau^2 - \kappa_1^2} + \frac{q_2\tau^4}{\tau^2 - \kappa_2^2} + \ldots\ldots\right\} \ldots (25).$$

From the manner in which this result has been obtained, it follows that $\kappa_1, \kappa_2 \ldots$ are in ascending order of magnitude. Now in the case of ordinary dispersion, μ increases as the period diminishes, whence we must have $\tau > \kappa_1$ and $< \kappa_2$; also the quantities q_2, q_3 must be inappreciable, and q_1 must be very slightly less than unity. Under these circumstances, we approximately obtain

$$\mu^2 = 1 + \frac{C}{\rho}\left\{q_1\kappa_1^2 - (1 - q_1)\tau^2 + \frac{q_1\kappa_1^4}{\tau^2} + \frac{q_1\kappa_1^6}{\tau^4} + \ldots\right\} \ldots (26).$$

If we omit the term $-(1 - q_1)\tau^2$, this expression is the same as Cauchy's dispersion formula, which agrees fairly well with experiment; Ketteler has however shown that for certain substances, the term $-(1 - q_1)\tau^2$ is required.

351. In order to apply (24) for the purpose of illustrating anomalous dispersion, it will be sufficient to confine our attention to the terms involving κ_1 and κ_2.

Differentiating (24) with respect to μ^2, we obtain

$$\frac{d\mu^2}{d\tau^2} = \frac{C}{\rho}\left\{q_1 + q_2 - \frac{q_1\kappa_1^4}{(\tau^2 - \kappa_1^2)^2} - \frac{q_2\kappa_2^4}{(\tau^2 - \kappa_2^2)^2} - 1\right\}.$$

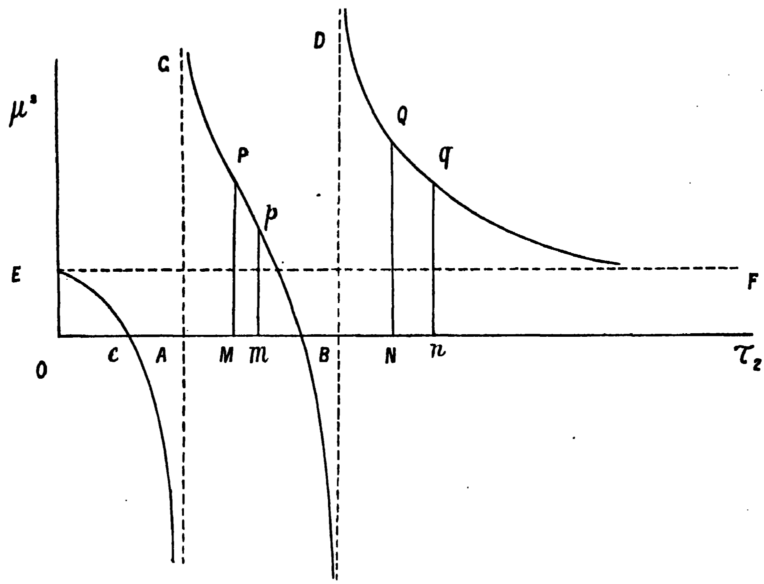
When $\tau = 0$, $d\mu^2/d\tau^2$ is negative; but when $\tau = \infty$, it will not be negative unless $q_1 + q_2 < 1$; since however μ increases as τ diminishes, we may suppose that the various quantities are so related, that $d\mu^2/d\tau^2$ is negative throughout that part of the spectrum, which is capable of being observed. And if we suppose q_1 is very nearly equal to unity, and $q_2 < 1 - q_1$ this will be the case for a considerable range of values of τ.

Let us further suppose that $\kappa_2 > \kappa_1$; then when τ is excessively small, μ^2 will be less than unity; as τ increases, μ^2 will diminish to zero, and will then become a negative quantity. When μ^2 is negative, the velocity of the waves will be imaginary, and consequently waves whose periods produce this result, are incapable of being propagated in the medium, and absorption will take place. When $\tau = \kappa_1$, $\mu^2 = -\infty$; and as soon as $\tau > \kappa_1$, μ^2 becomes a very large positive quantity, and regular refraction begins to take place. As τ further increases, μ^2 diminishes, until it vanishes and changes sign. A second absorption band accordingly commences, and continues until $\tau > \kappa_2$, when regular refraction begins again, and so continues until $\tau = \infty$.

352. The following figure will serve to give the reader a general idea of the value of μ^2. The abscissæ represent the values of τ^2, and the ordinates the values of μ^2. The dotted lines AC, BD, EF are the lines $\tau = \kappa_1$, $\tau = \kappa_2$, $\mu = 1$, and the upper parts of the curves Pp, Qq represent the visible portion of the spectrum produced by a prism filled with a substance, which produces anomalous dispersion, and has an absorption band in the green.

The portion Ee may be supposed to consist of waves whose periods are too short to be observed, then comes an absorption band, and beyond A a region of highly refrangible ultra-violet light commences. The line H in the spectrum may be supposed to commence at P; and a band of light accordingly becomes visible, which continues through the indigo to the blue, and in

which the violet is the most refracted and the blue the least. According to the figure, this ought to be followed by a region of



blue-green light, for which the index of refraction is *less* than unity; and we must therefore suppose, that on account of the narrowness of the region, or the faintness of the light, this region has either escaped observation, or is incapable of being detected without more powerful instruments. An absorption band then follows, and is succeeded by another band of more highly refracted light, corresponding to Qq, in which the red is the least, and the orange is the most refracted. Since the value of μ, when τ is slightly greater than κ_2, is large, it follows that the dispersion is anomalous; and we thus see why it is, that when there is an absorption band in the green, orange and red are more refracted than blue light.

353. If we compare these results with the table on p. 297, it will be seen, that they give a fairly satisfactory explanation of the anomalous dispersion produced by fuchsine and cyanine. The five absorption bands produced by permanganate of potash, could be explained by taking into account some of the terms in the value of μ^2, which have been omitted.

Von Helmholtz' Theory of Anomalous Dispersion.

354. The theory proposed by Von Helmholtz[1], is a theory relating to the mutual action of ether and matter, of somewhat the same character as Lord Rayleigh's theory of double refraction; but instead of following Von Helmholtz' method, we shall give the theory in a somewhat extended form[2].

Let u, v, w be the component displacements of the ether, and u_1, v_1, w_1 those of the matter. We shall suppose, that in vacuo the ether is a medium, whose motion is governed by the same equations as those of an elastic solid.

When ethereal waves pass through a material substance, the molecules of the matter will be displaced, and the matter will acquire potential energy. The proper form of the mathematical expression for this potential energy is a question of speculation; and the first hypothesis we shall make will be, that the molecular forces are proportional to the displacements of the matter, and consequently the potential energy W_3 of the matter will be of the form

$$W_3 = \tfrac{1}{2}(\mathfrak{A}u_1^2 + \mathfrak{B}v_1^2 + \mathfrak{C}w_1^2 + 2\mathfrak{A}'v_1w_1 + 2\mathfrak{B}'w_1u_1 + 2\mathfrak{C}'u_1v_1) \quad \ldots(1),$$

where $\mathfrak{A}$, $\mathfrak{B}$ ... are constants.

The second hypothesis is, that the potential energy of the system contains a term, which depends upon the relative displacements of ether and matter. This portion of the potential energy, which we shall denote by W_2, is supposed to arise from the mutual reaction of ether upon matter; and if we assume that the corresponding forces are linear functions of the relative displacements, W_2 will be a homogeneous quadratic function of the relative displacements, so that

$$\begin{aligned} W_2 = \tfrac{1}{2}\{&A(u-u_1)^2 + B(v-v_1)^2 + C(w-w_1)^2 + 2A'(v-v_1)(w-w_1) \\ &+ 2B'(w-w_1)(u-u_1) + 2C'(u-u_1)(v-v_1)\} \quad \ldots\ldots(2). \end{aligned}$$

The third part of the potential energy, which we shall denote by W_1, is the potential energy of the ether alone, and is of the same form as that of an elastic solid. The total potential energy W of the system will therefore be

$$W = W_1 + W_2 + W_3 \quad \ldots\ldots(3).$$

[1] *Pogg. Ann.* vol. CLIV. p. 582; *Wissen. Abhand.* vol. II. p. 213.

[2] *Proc. Lond. Math. Soc.* vol. XXIII. p. 4.

If ρ_1 be the density of the matter, its kinetic energy will be

$$T_2 = \tfrac{1}{2}\rho_1 (\dot{u}_1^2 + \dot{v}_1^2 + \dot{w}_1^2) \quad \text{.................}(4).$$

In order to introduce Lord Rayleigh's theory, we shall suppose that the effect of the matter upon the ether, is to cause the latter to behave as if its density were æolotropic, and the third hypothesis will therefore be, that the kinetic energy T_1 of the ether is

$$T_1 = \tfrac{1}{2}(P\dot{u}^2 + Q\dot{v}^2 + R\dot{w}^2) \quad \text{.................}(5).$$

355. The equations of motion of the system may now be deduced by the Principle of Least Action, viz.

$$\iiiint \delta (T_1 + T_2 - W_1 - W_2 - W_3)\, dxdydzdt = 0\,;$$

and will be found to be

$$P\frac{d^2u}{dt^2} = m\frac{d\delta}{dx} + n\nabla^2 u - \frac{dW_2}{du} \quad \text{..............}(6),$$

with two similar equations, and

$$\rho_1 \frac{d^2u_1}{dt^2} = -\frac{dW_2}{du_1} - \frac{dW_3}{du_1} \text{....................}(7)$$

with two similar equations.

When the medium is isotropic,

$$P = Q = R = \rho,$$

$$A = B = C = \alpha^2;\ A' = B' = C' = 0,$$

$$\mathfrak{A} = \mathfrak{B} = \mathfrak{C} = \beta^2;\ \mathfrak{A}' = \mathfrak{B}' = \mathfrak{C}' = 0,$$

and the equations of motion accordingly become

$$\rho\frac{d^2u}{dt^2} = m\frac{d\delta}{dx} + n\nabla^2 u - \alpha^2(u - u_1) \quad \text{...........}(8),$$

&c. &c.

$$\rho_1\frac{d^2u_1}{dt^2} = \alpha^2(u - u_1) - \beta^2 u_1 \quad \text{....................}(9),$$

&c. &c.

356. Von Helmholtz has introduced a viscous term into the equations of motion of the *matter*, which has been objected to by Lord Kelvin[1]. I think however that this term may be justified on two independent grounds. It is an experimental fact, that when vibrations are set up in any material substance, the vibrations are gradually damped by internal friction, and the motion ultimately

[1] *Lectures on Molecular Dynamics*, p. 149.

dies away. This state of things may be represented mathematically, by the introduction of a viscous term into the equations of motion of the matter. If however we do not wish to introduce the hypothesis of internal friction into our equations, the viscous term may still be accounted for. When a molecule of matter is set into vibration by ethereal waves, the physical characteristics of the motion will be much the same as those of the spherical pendulum vibrating in air, which has been discussed at the commencement of the present chapter; consequently the vibrations of the molecules will generate secondary waves in the ether, which will carry away into space the energy, which is communicated to the molecules, and the equations of motion of the latter will therefore contain a viscous term. In the place of (9), we shall accordingly assume that the equations of motion of the matter are of the form

$$\rho_1 \frac{d^2 u_1}{dt^2} + 2h \frac{du_1}{dt} = \alpha^2 (u - u_1) - \beta^2 u_1 \quad \text{.........(10).}$$

357. Let us now consider the propagation of waves of light, which are travelling through the medium in the positive direction of the axis of z, and let the displacements be parallel to the axis of x. Equation (8) now becomes

$$\rho \frac{d^2 u}{dt^2} = n \frac{d^2 u}{dz^2} + \alpha^2 (u_1 - u) \quad \text{..............(11).}$$

To solve (10) and (11) assume

$$u = A\epsilon^{-qz + 2\iota\pi/\tau \,.\, (z/V - t)},$$

$$u_1 = A_1 \epsilon^{-qz + 2\iota\pi/\tau \,.\, (z/V - t)};$$

then (11) and (10) become,

$$\left\{\frac{4\pi^2 \rho}{\tau^2} + n\left(q - \frac{2\iota\pi}{V\tau}\right)^2 - \alpha^2\right\} A = -\alpha^2 A_1 \text{.........(12),}$$

$$\left\{\frac{4\pi^2 \rho_1}{\tau^2} - \beta^2 - \alpha^2 + \frac{4\iota\pi h}{\tau}\right\} A_1 = -\alpha^2 A \quad \text{.........(13).}$$

Now if κ be the free period of the matter vibrations,

$$\beta^2 = \frac{4\pi^2 \rho_1}{\kappa^2} + \frac{h^2}{\rho_1};$$

whence from (13), we obtain

$$\frac{A_1}{A} = -\frac{\alpha^2}{4\pi^2 \rho_1 (\tau^{-2} - \kappa^{-2}) - \alpha^2 - h^2/\rho_1 + 4\iota\pi h/\tau} \text{.....(14).}$$

From the problem of the spherical pendulum we infer, that the amplitudes of the molecules of matter are very much smaller than those of the ether; and accordingly α^2 must be very small compared with ρ_1 the density of the matter. If however

$$\frac{1}{\tau^2} = \frac{1}{\kappa^2} + \frac{\alpha^2}{4\pi^2\rho_1},$$

in which case the period of the waves of light would be sensibly equal to the free period of the matter, A_1 would be infinite in the absence of friction. This conclusion is contradicted by experience, and necessitates the introduction of a viscous term.

358. Let D denote the real part of the denominator of (14); then if we eliminate A, A_1 from (12) and (13), and equate the real and imaginary parts, we shall obtain

$$\left(\frac{4\pi^2\rho}{\tau^2} + nq^2 - \frac{4\pi^2 n}{V^2\tau^2} - \alpha^2\right) D + \frac{16\pi^2 nhq}{V\tau^2} = \alpha^4 \quad \text{......(15)},$$

$$\left(\frac{4\pi^2\rho}{\tau^2} + nq^2 - \frac{4\pi^2 n}{V^2\tau^2} - \alpha^2\right) h - \frac{nqD}{V} = 0 \quad \text{...........(16)}.$$

Now q is the coefficient of absorption of the medium; and since the absorption is slight for most transparent substances, it follows that q, and therefore h, must be small.

Let P denote the coefficient of h in (16), then eliminating h between (15) and (16), we obtain

$$P^2D - \alpha^4 P + \frac{16\pi^2 n^2 q^2 D}{V^2\tau^2} = 0,$$

whence
$$P = \tfrac{1}{2}\left\{\alpha^4 + \left(\alpha^8 - \frac{64\pi^2 n^2 q^2 D^2}{V^2\tau^2}\right)^{\frac{1}{2}}\right\} D^{-1},$$

the positive sign being taken, because $P = \alpha^4/D$, when $q = 0$.

Expanding the quantity under the radical, we obtain

$$P = \frac{\alpha^4}{D} - \frac{16\pi^2 n^2 q^2 D}{\alpha^4 V^2 \tau^2} \quad \text{.................(17)}.$$

Now if ρ_0 be the density of the ether in vacuo, $n/V^2 = \rho_0\mu^2$, where μ is the index of refraction; whence (17) becomes

$$\mu^2 = \left(1 - \frac{4nq^2}{\alpha^4}\right)^{-1} \left\{\frac{\rho}{\rho_0} + \frac{nq^2\tau^2}{4\pi^2\rho_0} - \frac{\alpha^2\tau^2}{4\pi^2\rho_0}\left[1 + \frac{\alpha^2}{4\pi^2\rho_1(\tau^{-2} - \kappa^{-2}) - \alpha^2 - h^2/\rho_1}\right]\right\} \quad \text{..... (18)}.$$

359. Since the absorbing power of transparent media is small, the value of the index of refraction will scarcely be affected thereby; we may therefore as a sufficient approximation put $q=0$, and (18) becomes

$$\mu^2 = \frac{\rho}{\rho_0} - \frac{\alpha^2\tau^2}{4\pi^2\rho_0}\left\{1 + \frac{\alpha^2\kappa^2\tau^2}{4\pi^2\rho_1(\kappa^2-\tau^2) - \alpha^2\kappa^2\tau^2}\right\} \quad \text{......(19).}$$

Let $$P = \alpha^2/4\pi^2\rho_0, \quad Q = \alpha^2/4\pi^2\rho_1,$$

then $$\frac{d\mu^2}{d\tau^2} = -\frac{P}{(\kappa^2-\tau^2-Q\kappa^2\tau^2)^2}\{(\kappa^2-\tau^2)^2 + Q\kappa^2\tau^4\} \quad \text{...(20),}$$

whence $d\mu^2/d\tau^2$ is *negative; it therefore follows that μ^2 decreases as τ^2 increases.*

Since ρ is the density of ether when loaded with matter, it follows that $\rho > \rho_0$; hence when $\tau = 0$, $\mu > 1$. As τ increases, μ^2 diminishes to unity; it then becomes less than unity, until τ attains a value τ_3, which makes $\mu = 0$. When $\tau > \tau_3$, μ^2 is negative; and consequently at this point an absorption band commences, which continues until $\tau = \tau_2$, where τ_2 is the value of τ which makes the denominator vanish. When $\tau = \tau_2$, $\mu^2 = -\infty$; and when $\tau > \tau_2$, μ^2 is a very large positive quantity, and regular refraction begins again. As τ still further increases, μ^2 continues to diminish, until τ attains a value τ_1, such that μ^2 is again zero; when $\tau > \tau_1$, μ^2 becomes negative, and remains so for all greater values of τ.

The medium is therefore absolutely opaque to waves whose periods are greater than τ_1; it is transparent for waves whose periods are less than τ_1 and greater than τ_2; it is opaque for waves whose periods lie between τ_2 and τ_3, and is transparent for waves of shorter period.

If we now suppose that τ_2 corresponds to the double sodium line *D*, whilst τ_3 corresponds to the hydrogen line *F*, we shall obtain a mechanical representation of a medium, which has an absorption band in the green; also the dispersion is anomalous, since the value of μ^2 when τ is a little greater than τ_2, is greater than it is when τ is a little less than τ_3. A medium of this kind accordingly represents a substance such as fuchsine, which has an absorption band in the green, and produces anomalous dispersion.

360. To explain ordinary dispersion, we shall suppose that τ is greater than κ; then if we put

$$b^2 = \kappa^2/(1 + Q\kappa^2),$$

(19) may be written

$$\mu^2 = \frac{\rho}{\rho_0} + PQb^4 - \frac{Pb^2\tau^2}{\kappa^2} + PQb^4\left(\frac{b^2}{\tau^2} + \frac{b^4}{\tau^4} + \dots\right).$$

With the exception of the term involving τ^2, this value of μ^2 is of the same form as Cauchy's formula

$$\mu^2 = a + c/\lambda^2 + d/\lambda^4 + \dots.$$

Ketteler[1] has however shown that the term $Pb^2\tau^2/\kappa^2$ is required to explain the dispersion produced by certain substances.

361. When there are several absorption bands, a molecule of a more complicated character is required; and it has been suggested by Von Helmholtz, that a theory might be constructed by a hypothesis, which practically amounts to assuming that W_2 and W_3 consist of a series of terms of the form

$$W_2 = \tfrac{1}{2}\Sigma\alpha_s^2\{(u - u_s)^2 + (v - v_s)^2 + (w - w_s)^2\}$$
$$W_3 = \tfrac{1}{2}\Sigma\beta_s^2(u_s^2 + v_s^2 + w_s^2).$$

Selective Reflection.

362. We must now consider the reflection of light at the surface of a medium, which produces anomalous dispersion.

In forming the equations of motion by means of the Principle of Least Action, we observe that there are no surface integral terms, which arise from W_2 and W_3; it therefore follows, that the boundary conditions at the common surface of two different media, are unaffected by the presence of the terms depending on the action of the matter. These conditions will therefore be, continuity of the displacements and stresses arising from the action of the ether. With regard to the physical properties of the ether, I shall provisionally adopt the hypothesis of Lord Kelvin, that the latter is to be treated as an elastic medium, whose resistance to compression is a negative quantity, the numerical value of which is slightly less than $\frac{4}{3}$rds of the rigidity.

Under these circumstances, the intensities of the reflected and refracted light will be given by Fresnel's formulæ, and so long as $\mu > 1$, the reflection takes place in the same manner as from glass. If however the incident light is white, and κ lies within the visible spectrum, say between D and F, it follows that for certain rays of the spectrum $\mu < 1$, in which case there will be a critical angle.

[1] Treatise on *Theoretische Optik.*

Hence those rays, for which $\mu < \sin i$, will be totally reflected with a change of phase. There will also be another set of rays, for which μ^2 is a negative quantity, and we shall now show that these rays are also totally reflected with a change of phase.

363. Let $\mu^2 = -\nu^2$, then

$$\sin i = \iota\nu \sin r, \qquad \cos r = \nu^{-1}(\nu^2 + \sin^2 i)^{\frac{1}{2}}.$$

Now when the light is polarized in the plane of incidence

$$\begin{aligned} A' &= -\frac{A \sin(i-r)}{\sin(i+r)} \\ &= A\frac{\cos i - \iota(\nu^2+\sin^2 i)^{\frac{1}{2}}}{\cos i + \iota(\nu^2+\sin^2 i)^{\frac{1}{2}}} \\ &= A\epsilon^{-2\iota\pi f/\lambda}, \end{aligned}$$

where $$\tan \pi f/\lambda = (\nu^2 + \sin^2 i)^{\frac{1}{2}}/\nu \qquad (21),$$

which shows that the reflection is total, and that there is a change of phase, whose value is determined by (21).

Similarly when the light is polarized perpendicularly to the plane of incidence,

$$\begin{aligned} \frac{B'}{B} &= \frac{\tan(i-r)}{\tan(i+r)} \\ &= \epsilon^{2\iota\pi f_1/\lambda}, \end{aligned}$$

where $$\tan \pi f_1/\lambda = \frac{\nu^2}{(\nu^2 + \sin^2 i)^{\frac{1}{2}}} \qquad (22).$$

which shows that the reflection is total, and is accompanied by a change of phase which is determined by equation (22).

Since the change of phase is different, according as the light is polarized in or perpendicularly to the plane of incidence, it follows that the reflected light will be elliptically polarized.

364. We must now enquire, how far these results will explain the selective reflection of substances, which produce anomalous dispersion.

Let τ_3 be the least value of τ for which $\mu = 0$, and let τ_2 be the value for which $\mu^2 = \pm\infty$. Then if we suppose that τ_2 and τ_3 respectively correspond to the lines D and F, it follows that there will be an absorption band in the green; accordingly the substance will transmit the blue and red rays, and some of the yellow, hence the colour of the transmitted light will be red or reddish blue. Since μ is a real quantity for these rays, the latter will be reflected in the ordinary manner; but the green rays, for which μ is

imaginary, will be totally reflected with a change of phase; hence the colour of the reflected light will be green, and the light will be elliptically polarized.

In fuchsine the order of colours going up the spectrum is F, G, H, then there is an absorption band, and then come the lines A, B C, D. The absorption is very strong between D and F; accordingly fuchsine is opaque to the green portion of the spectrum, and ought to reflect green light strongly. This agrees with observation. Now although fuchsine is not absolutely opaque to blue and yellow light, the prevailing colour of the transmitted light is rose; accordingly this substance must possess a tendency to transmit red light to a far greater extent than blue or yellow, and consequently ought to reflect the latter colours more strongly than the former one. This is also borne out by observation, since the reflected light is green.

365. Let us now suppose, that the incident light is common light; and that the reflected light is examined through a Nicol's prism, whose principal section is parallel to the plane of incidence. Then the vibrations perpendicular to the plane of incidence cannot get through the Nicol, and may be therefore left out of account. Now the reflected light consists of three portions; (i) the portion for which μ^2 is negative, and which begins a little above D and ends a little below F; (ii) a certain portion for which μ^2 is positive, and less than unity, and which lies in the neighbourhood of F; (iii) the portion for which $\mu^2 > 1$, which is regularly reflected, and constitutes the remaining portion of the spectrum. The first portion is by far the most intense, since the reflection is total; the third portion is the least intense; whilst of the second, for which $\mu < 1$, those rays for which the critical angle is less than the angle of incidence, will be totally reflected, and those for which it is greater, will be regularly reflected. Now if the angle of incidence is nearly equal to the polarizing angle, it follows that the yellow portion of the incident light, most of which is regularly reflected, will be polarized in the plane of incidence by reflection, and will therefore be unable to get through the Nicol; but the green, and also a portion of the blue in the neighbourhood of F, will get through. The colour of the light, when viewed through a Nicol, will accordingly change from a green to a greenish blue, owing to the absence of the yellow light.

CHAPTER XVIII.

METALLIC REFLECTION.

366. THE leading experimental facts connected with metallic reflection may be classified as follows.

(i) *Metals are exceedingly opaque to light, but at the same time reflect a very large proportion of the incident light.*

(ii) *When plane polarized light is incident upon a polished metallic surface, the reflected light is always elliptically polarized, unless the incident light is polarized in or perpendicularly to the plane of incidence, in which case the reflected light is plane polarized.*

(iii) *Metals do not possess a polarizing angle, but there is a certain angle of incidence, for which the intensity of light polarized perpendicularly to the plane of incidence is a minimum.*

(iv) *When the incident light is circularly polarized, there is a certain angle of incidence, for which the reflected light is plane polarized.*

Whatever the character of the incident light may be, it can always be resolved into two components, which are respectively in and perpendicular to the plane of incidence; and the above experimental results show, that metallic reflection produces a change of phase in one or both of these components.

367. The angle of incidence, for which circularly polarized light is converted into plane polarized light, is called *the principal incidence;* and the azimuth of the plane of polarization of the reflected light, is called *the principal azimuth.* The principal azimuth is usually measured from the plane of incidence

towards the right hand of an observer, who is looking at the point of incidence along the reflected ray. Since the course of a ray may be supposed to be reversed, it follows that if light polarized in the principal azimuth, is reflected at the principal incidence, the reflected light will be circularly polarized.

368. The values of the principal incidence and azimuth depend not only upon the particular metal of which the reflector consists, but also upon the transparent medium in contact with it; and it has been found by experiment, that the principal incidence diminishes, whilst the principal azimuth increases with the increase of the index of refraction of the medium in contact with the metallic reflector.

369. Although a plate of metal of sensible thickness is opaque, yet a very thin film of metal is semi-transparent; and if white light be incident upon the film, the transmitted light is frequently coloured. Thus, if sunlight is passed through a piece of gold leaf, the transmitted light is green. The experiments of Quincke[1] show, that the phases of both components of the refracted light are accelerated by transmission; and Sir John Conroy[2] has shown, that when light is reflected from a thin metallic film, the principal incidence and azimuth both increase with the thickness of the film.

That a thin film should reflect light differently from a thick plate is to be expected. For since thin films are semi-transparent, the wave penetrates a sufficient distance to be reflected from the posterior surface; whilst when the plate is thick, no second reflection takes place, owing to the refracted wave being extinguished before arriving at the posterior surface. If a perfectly satisfactory theory of metallic reflection existed, there would be no theoretical difficulty in explaining the peculiarities connected with reflection from, and transmission through, thin metallic films; all that would be necessary would be, to take into account the successive reflections and refractions from both surfaces of the film, in the same way as is done in the ordinary theory of the Colours of Thin Transparent Plates.

370. The opacity of metals can be partially explained by supposing, that the index of refraction is a complex quantity; but

[1] *Pogg. Ann.* vol. CXXIX.

[2] *Proc. Roy. Soc.* vol. XXXI. p. 500.

as this statement is ambiguous, we shall proceed to consider it carefully.

Let the axis of x be the normal to a metallic reflector in contact with air; and let the displacements of the incident and refracted waves be

$$\left.\begin{aligned} w &= A\epsilon^{\frac{2\iota\pi}{V\tau}(lx+my-Vt)} \\ w_1 &= A_1\epsilon^{\frac{2\iota\pi}{V_1\tau}(l_1x+m_1y-V_1t)} \end{aligned}\right\} \qquad (1).$$

Since the coefficients of y must be the same in the two waves, we must have

$$\frac{V}{V_1} = \frac{\sin i}{\sin r} = \mu \text{ (say)} \qquad (2).$$

If the second medium were transparent, V_1 would be a real quantity, and consequently μ would be real; but in a metal, there is properly speaking no refracted wave, and therefore V_1, and consequently μ, cannot be real. We must therefore suppose, that μ is complex.

For these reasons it is often said, that the index of refraction of metals is a complex quantity. The expression is not however very happily chosen, since there is no such thing as an index of refraction in the case of metals. If however we regard the index of refraction as a convenient name for the mathematical quantity, which is defined by (2), there will be no danger of any ambiguity.

Putting $\mu = R\epsilon^{\iota\alpha}$, we obtain

$$V_1 = \frac{V}{R}(\cos\alpha - \iota\sin\alpha),$$

and therefore, when the incidence is normal, so that $l = l_1 = -1$,

$$w_1 = A\epsilon^{2\pi/\lambda \,.\, Rx\sin\alpha}\cos\frac{2\pi}{\lambda}(Rx\cos\alpha + Vt),$$

where λ is the wave-length in the first medium, which is supposed to be transparent.

Now x is negative in the second medium; accordingly $\sin\alpha$ must be positive, otherwise the amplitude would increase as x increases. Hence the amplitude diminishes very rapidly as the distance from the surface of separation increases, and at a distance of a few wave-lengths in air, the refracted wave becomes insensible.

Since $$\mu^2 = R^2(\cos 2\alpha + \iota \sin 2\alpha),$$
and $\sin\alpha$ must be positive, it follows that α must lie between 0 and $\frac{1}{2}\pi$. Hence μ^2 must be a complex quantity, whose imaginary part must be positive, but whose real part may be either positive or negative. We shall presently show, that there are reasons for thinking that for certain metals the real part of μ^2 must be negative, in which case α must lie between $\frac{1}{4}\pi$ and $\frac{1}{2}\pi$.

371. Theories of metallic reflection have been proposed by MacCullagh[1], Cauchy[2] and others; and although these theories in their original form cannot be said to stand on a satisfactory physical basis, yet the formulæ of Cauchy furnish results, which agree fairly well with experiment, and may therefore be regarded as an empirical representation of the facts. We have shown in the previous chapter, that it is possible to construct a dynamical theory, such that for certain rays of the spectrum, μ^2 shall be a real negative quantity; but from § 370, it follows that in the case of a metal, μ^2 must be a complex quantity, whose imaginary part must be positive. Following Eisenlohr[3], we shall first show how Cauchy's formulæ may be deduced by transforming Fresnel's formulæ for transparent media; and shall afterwards discuss a dynamical theory, by means of which this transformation may be justified.

Cauchy's Theory.

372. When the incident light is polarized in the plane of incidence, Fresnel's formulæ for the amplitudes of the reflected and refracted light are

$$A' = -\frac{\sin(i-r)}{\sin(i+r)} \quad\text{..........................(3)},$$

$$A_1 = \frac{2\sin r\cos i}{\sin(i+r)} \quad\text{..........................(4)},$$

in which the amplitude of the incident light is taken as unity. We have now to transform these formulæ, by supposing that μ is a complex quantity of the form $R\epsilon^{\iota\alpha}$.

[1] *Proc. Roy. Ir. Acad.* vol. I. p. 2.
[2] *C. R.* 1838 and 1839.
[3] *Pogg. Ann.* vol. CIV. p. 368.

Equation (3) may be written

$$A' = \frac{\cos i - \mu \cos r}{\cos i + \mu \cos r} \text{ ....(5),}$$

where $$\cos r = (1 - \mu^{-2} \sin^2 i)^{\frac{1}{2}} \text{ ....(6).}$$

Since μ is complex, it follows that $\cos r$ is complex, and we shall therefore put it equal to $c\epsilon^{\iota u}$; whence we obtain from (6)

$$\left.\begin{aligned} c^2 \cos 2u &= 1 - R^{-2} \cos 2\alpha \sin^2 i \\ c^2 \sin 2u &= R^{-2} \sin 2\alpha \sin^2 i \end{aligned}\right\} \text{ ....(7),}$$

which determine c and u in terms of R, α and i. Equation (5) now becomes

$$\begin{aligned} A' &= \frac{\cos i - Rc\epsilon^{\iota(\alpha+u)}}{\cos i + Rc\epsilon^{\iota(\alpha+u)}} \\ &= -\frac{R^2c^2 - \cos^2 i + 2\iota Rc \cos i \sin (\alpha + u)}{R^2c^2 + \cos^2 i + 2Rc \cos i \cos (\alpha + u)} \\ &= -\mathfrak{A}\epsilon^{\frac{2\iota\pi e}{\lambda}} \text{ ....(8),} \end{aligned}$$

where $$\mathfrak{A}^2 = \frac{R^2c^2 + \cos^2 i - 2Rc \cos i \cos (\alpha + u)}{R^2c^2 + \cos^2 i + 2Rc \cos i \cos (\alpha + u)} \text{ ....(9),}$$

$$\tan \frac{2\pi e}{\lambda} = \frac{2Rc \cos i \sin (\alpha + u)}{R^2c^2 - \cos^2 i} \text{ ....(10),}$$

whence the reflected wave, which is the real part of

$$-\mathfrak{A}\epsilon^{\frac{2\iota\pi}{\lambda}(x \cos i + y \sin i - Vt + e)}$$

is $$-\mathfrak{A} \cos \frac{2\pi}{\lambda} (x \cos i + y \sin i - Vt + e),$$

which shows that reflection is accompanied by a change of phase, whose value is given by (10).

If we introduce a new angle f, such that

$$\begin{aligned} \cot f &= \frac{2Rc \cos i \cos (\alpha + u)}{R^2c^2 + \cos^2 i} \\ &= \cos (\alpha + u) \sin 2 \left(\tan^{-1} \frac{\cos i}{Rc}\right) \text{ ....(11),} \end{aligned}$$

(9) and (10) become

$$\mathfrak{A}^2 = \tan (f - \tfrac{1}{4}\pi) \text{ ....(12),}$$

$$\tan \frac{2\pi e}{\lambda} = \sin (\alpha + u) \tan 2 \left(\tan^{-1} \frac{\cos i}{Rc}\right) \text{ ....(13).}$$

Equations (11), (12) and (13) are Cauchy's formulæ for light polarized in the plane of incidence.

373. To find what the refracted wave becomes, we have from (4)

$$A_1 = \frac{2\cos i}{\cos i + \mu \cos r}$$

$$= \frac{2\cos i\,\{Rc\cos(\alpha+u) + \cos i - \iota Rc\sin(\alpha+u)\}}{R^2c^2 + \cos^2 i + 2Rc\cos i\cos(\alpha+u)}$$

$$= \mathfrak{A}_1 \epsilon^{-\frac{2\iota\pi e_1}{\lambda}},$$

where $$\mathfrak{A}_1^2 = \frac{4\cos^2 i}{R^2c^2 + \cos^2 i + 2Rc\cos i\cos(\alpha+u)},$$

$$\tan\frac{2\pi e_1}{\lambda} = \frac{Rc\sin(\alpha+u)}{Rc\cos(\alpha+u) + \cos i},$$

whence the refracted wave, which is the real part of

$$\mathfrak{A}_1 \epsilon^{\frac{2\iota\pi}{\lambda}(-\mu x\cos r + y\sin i - Vt - e_1)},$$

is

$$\mathfrak{A}_1 \epsilon^{\frac{2\pi}{\lambda}Rcx\sin(\alpha+u)} \cos\frac{2\pi}{\lambda}\{Rcx\cos(\alpha+u) - y\sin i + Vt + e_1\} \quad \text{...(14).}$$

We have already shown that α is a positive quantity, whence it follows from (7), that u must also be positive; accordingly since x is negative in the second medium, the exponential factor diminishes very rapidly, and the refracted wave becomes insensible at a distance of a few wave-lengths from the surface of the reflecting surface. It also follows from (14), that the velocity of the refracted wave in the metal at normal incidence is equal to

$$V/R\cos\alpha.$$

374. When the incident light is polarized perpendicularly to the plane of incidence, Fresnel's formulæ become

$$B' = \frac{\tan(i-r)}{\tan(i+r)},$$

$$B_1 = \frac{2\sin r\cos i}{\sin(i+r)\cos(i-r)};$$

whence $$B' = \frac{\mu\cos i - \cos r}{\mu\cos i + \cos r}$$

$$= \frac{R\cos i - c\epsilon^{-\iota(\alpha-u)}}{R\cos i + c\epsilon^{-\iota(\alpha-u)}}$$

$$= \frac{R^2\cos^2 i - c^2 + 2\iota Rc\cos i\sin(\alpha-u)}{R^2\cos^2 i + c^2 + 2Rc\cos i\cos(\alpha-u)}$$

$$= \mathfrak{B}\epsilon^{\frac{2\iota\pi e'}{\lambda}},$$

where $$\mathfrak{B}^2 = \frac{R^2\cos^2 i + c^2 - 2Rc\cos i\cos(\alpha-u)}{R^2\cos^2 i + c^2 + 2Rc\cos i\cos(\alpha-u)} \quad \text{..........(15),}$$

$$\tan \frac{2\pi e'}{\lambda} = \frac{2Rc \cos i \sin (\alpha - u)}{R^2 \cos^2 i - c^2} \quad \text{.........................(16)},$$

whence the reflected wave is

$$\mathfrak{B} \cos \frac{2\pi}{\lambda} (x \cos i + y \sin i - Vt + e'),$$

which shows that, in this case also, metallic reflection is accompanied by a change of phase, whose value is given by (16). Since the changes of phase are different, according as the incident light is polarized in or perpendicularly to the plane of incidence, it follows that when the incident light is polarized in any azimuth, the reflected light will usually be elliptically polarized.

If we introduce an angle g, such that

$$\cot g = \frac{2Rc \cos i \cos (\alpha - u)}{R^2 \cos^2 i + c^2}$$

$$= \cos (\alpha - u) \sin 2 (\tan^{-1} c/R \cos i) \quad \text{.........(17)},$$

(15) and (16) become

$$\mathfrak{B}^2 = \tan (g - \tfrac{1}{4}\pi) \quad \text{...............................(18)},$$

$$\tan \frac{2\pi e'}{\lambda} = \sin (\alpha - u) \tan 2 (\tan^{-1} c/R \cos i) \quad \text{........(19)}.$$

Equations (17), (18) and (19) are Cauchy's formulæ for light polarized perpendicularly to the plane of incidence.

From these results it follows, that reflection from metals presents characteristics similar to total reflection from glass in contact with air, and also to the selective reflection produced by aniline dyes and other colouring materials.

375. We shall now obtain expressions for the ratio of the amplitudes, and the difference of the changes of phase; but for the purpose of greater generality, we shall suppose the incident light to be elliptically polarized.

Let B and A be the amplitudes of the two components of the incident vibrations, in and perpendicularly to the plane of incidence; then by §§ 372 and 374, we have

$$\mathfrak{A} \epsilon^{\frac{2\iota\pi e}{\lambda}} = \frac{A \sin (i - r)}{\sin (i + r)},$$

$$\mathfrak{B} \epsilon^{\frac{2\iota\pi e'}{\lambda}} = \frac{B \tan (i - r)}{\tan (i + r)},$$

whence

$$\frac{\mathfrak{B}}{\mathfrak{A}} \epsilon^{\frac{2\iota\pi}{\lambda}(e' - e)} = \frac{B \cos (i + r)}{A \cos (i - r)} \quad \text{......................(20)}.$$

We must now transform the right-hand side of (20), in the same manner as we have done in the case of Fresnel's formulæ; and we shall find that

$$\frac{\mathfrak{B}}{\mathfrak{A}}\epsilon^{\frac{2\iota\pi}{\lambda}(e'-e)}=\frac{B}{A}\cdot\frac{Rc\cos i\epsilon^{\iota(\alpha+u)}-\sin^2 i}{Rc\cos i\epsilon^{\iota(\alpha+u)}+\sin^2 i}$$

$$=\frac{B}{A}\cdot\frac{R^2c^2\cos^2 i-\sin^4 i+2\iota Rc\cos i\sin^2 i\sin(\alpha+u)}{R^2c^2\cos^2 i+\sin^4 i+2Rc\cos i\sin^2 i\cos(\alpha+u)},$$

whence

$$\frac{\mathfrak{B}^2}{\mathfrak{A}^2}=\frac{B^2}{A^2}\cdot\frac{R^2c^2\cos^2 i+\sin^4 i-2Rc\cos i\sin^2 i\cos(\alpha+u)}{R^2c^2\cos^2 i+\sin^4 i+2Rc\cos i\sin^2 i\cos(\alpha+u)}\;\ldots\ldots(21),$$

and $$\tan\frac{2\pi}{\lambda}(e'-e)=\frac{2Rc\cos i\sin^2 i\sin(\alpha+u)}{R^2c^2\cos^2 i-\sin^4 i}\;\ldots\ldots(22).$$

If the incident light is circularly polarized, $A=B$; also if the angle of incidence is such that

$$Rc\cos i=\sin^2 i\;\ldots\ldots\ldots\ldots(23),$$

it follows from (22), that

$$e'=e+\tfrac{1}{4}\lambda,$$

whence the reflected light is plane polarized. If β be the azimuth of the plane of polarization, we obtain from (21) and (23)

$$\tan^2\beta=\frac{1-\cos(\alpha+u)}{1+\cos(\alpha+u)}=\tan^2\tfrac{1}{2}(\alpha+u),$$

whence $$\beta=\tfrac{1}{2}(\alpha+u)\;\ldots\ldots\ldots\ldots(24).$$

We have therefore established the fourth experimental law, which is enunciated in § 366, by means of Cauchy's theory. Accordingly (23) and (24) combined with (7) determine the *principal incidence*, and the *principal azimuth*. It also follows from the formulæ, that if light which is plane polarized in the principal azimuth be incident at the principal incidence, the reflected light will be circularly polarized. We shall denote the principal incidence by I.

376. These results enable us to calculate the constants R and α.

Let the azimuth of the plane of polarization be defined to be the angle, which this plane makes with the plane of incidence, measured to the right hand of an observer who is looking at the reflected light through an analyser. Let the light polarized at an azimuth $\frac{1}{4}\pi$ undergo two reflections at two parallel plates of

metal; and let the angle of incidence be equal to the principal incidence.

After undergoing two reflections, the component displacements perpendicular to and in the plane of incidence will be

$$\xi\sqrt{2} = \mathfrak{A}^2 \cos\frac{2\pi}{\lambda}(x \sin I + y \sin I - Vt + 2e),$$

$$\eta\sqrt{2} = \mathfrak{B}^2 \cos\frac{2\pi}{\lambda}(x \sin I + y \sin I - Vt + 2e + \tfrac{1}{2}\lambda);$$

whence the light which has been twice reflected, will be plane polarized at an azimuth χ, where

$$\tan\chi = \eta/\xi = -\mathfrak{B}^2/\mathfrak{A}^2 = -\tan^2\beta \text{ ....(25).}$$

Now the principal incidence I, and the azimuth χ, can be determined experimentally, whence by (23) and (25) the values of Rc and β can be found; accordingly from (24) and (7) the values of R and α can be calculated.

377. Jamin[1] has tested Cauchy's formulæ for the intensities of light polarized in and perpendicularly to the plane of incidence, and has found that they agree fairly well with experiment. His method of procedure was as follows. The quantities R and α were first determined by experiment, and the amplitudes $\mathfrak{A}$ and $\mathfrak{B}$ were then calculated for different angles of incidence by means of (9) and (15). This process gives the theoretical values of the ratio of the intensities of the incident and reflected light.

To obtain the values of the intensities by experiment, a plate of glass and metal were placed side by side, so as to be accurately in the same plane. A pencil of light was then allowed to fall on the compound reflecting surface, so that part was reflected by the glass, and part by the metal; and the two portions of the reflected light were passed through a doubly refracting prism, whose principal section was inclined at an angle γ to the plane of incidence.

The light on emerging from the prism, thus consisted of four images, two of which were produced by reflection from the metal, and the other two by reflection from the glass. Let $\mathfrak{A}^2$, $\mathfrak{A}'^2$ be the intensities of the light reflected from the metal and the glass, when the incident light is polarized *in* the plane of incidence;

[1] *Ann. de Chimie et de Physique*, Vol. XIX. p. 206.

then on emerging from the doubly refracting prism, the intensities of the ordinary and extraordinary images will be

	Metal	Glass
O......	$\mathfrak{A}^2 \cos^2 \gamma$,	$\mathfrak{A}'^2 \cos^2 \gamma$;
E......	$\mathfrak{A}^2 \sin^2 \gamma$,	$\mathfrak{A}'^2 \sin^2 \gamma$.

For a certain value of γ, the intensity of the ordinary image of the metal, will be equal to the extraordinary image of the glass. For this value,

$$\mathfrak{A}^2 \cos^2 \gamma = \mathfrak{A}'^2 \sin^2 \gamma;$$

whence remembering the value of $\mathfrak{A}'$, we obtain

$$\mathfrak{A}^2 = \tan^2 \gamma \frac{\sin^2 (i-r)}{\sin^2 (i+r)} \quad \text{.................(26).}$$

The value of γ is determined, by observing the angle at which the intensities of the two images become equal, and thence $\mathfrak{A}^2$ can be found by (26).

If the incident light had been polarized perpendicularly to the plane of incidence, we should have had

$$\mathfrak{B}^2 = \tan^2 \gamma \frac{\tan^2 (i-r)}{\tan^2 (i+r)} \quad \text{.................(27),}$$

but inasmuch as the intensity of the light reflected from the glass, is exceedingly small in the neighbourhood of the polarizing angle, accurate results cannot be obtained, when i is nearly equal to this angle.

The following table for steel, which is taken from Jamin's paper, shows how far theory and experiment agree.

Steel. Principal incidence 76°.

Angle of Incidence	$\mathfrak{A}$		$\mathfrak{B}$	
	Observed	Calculated	Observed	Calculated
85°	·951	·977	·719	·709
75°	·946	·932	·566	·563
65°	·898	·892	·627	·599
55°	·869	·856	...	...
45°	·818	·827	·689	·701
35°	·800	·804	·741	·717
25°	·791	·787	·769	·751

378. Jamin also made experiments upon the difference of the changes of phase of the two components; and arrived at the following laws.

(i) *The wave which is polarized perpendicularly to the plane of incidence, is more retarded than that which is polarized in the plane of incidence.*

(ii) *The difference of phase is zero at normal incidence, and increases up to grazing incidence.*

From (22), we see that

when $i = 0$, $e' - e = 0$;

when $i = I$, $e' - e = \frac{1}{4}\lambda$;

when $i = \frac{1}{2}\pi$, $e' - e = \frac{1}{2}\lambda$;

so that $e' > e$, and their difference gradually increases from $i = 0$ to $i = \frac{1}{2}\pi$.

379. Experiments on the difference between the changes of phase were made by Jamin by the method of multiple reflections. When light polarized in any azimuth is reflected m times from two parallel metallic reflectors, the difference of phase of the resulting light is $m(e' - e)$; and if this quantity is equal to a multiple of $\frac{1}{2}\lambda$, the resulting light will be plane polarized. This will be the case, when the angle of incidence is such that

$$e' - e = n\lambda/2m \ldots\ldots\ldots\ldots\ldots\ldots\ldots (28),$$

where n is equal to 1, 2, ... $m - 1$. The least angle of incidence at which this can happen, is given by $e' - e = \lambda/2m$; and the greatest by $e' - e = (m - 1)\lambda/2m$. Hence there are altogether $m - 1$ angles of incidence. Now these angles of incidence can be observed, and the resulting differences between the changes of phase calculated from (22), and compared with (28), and the two results ought to agree. For example, let three reflections take place, and let i_1, i_2, the least and greatest angles, at which polarization is re-established, be observed; then if we substitute the values of i_1, i_2 in (22), the resulting values of $e' - e$ corresponding to i_1, i_2 ought to be $\frac{1}{6}\lambda$ and $\frac{1}{3}\lambda$. We have thus a method of testing the formulæ for the difference between the changes of phase experimentally; and the experiments of Jamin show, that there is a fair agreement between experiment and theory.

380. It has been stated in § 368, that the principal incidence and principal azimuth, depend not only upon the nature of the metal, but also upon the medium in contact with it. The values

of these angles have been determined experimentally by Quincke[1] for silver, and by Sir John Conroy[2] for gold and silver, when certain other media are substituted for air. The following table shows the results obtained by the latter, when the incident light was red.

Medium	Principal Incidence	Principal Azimuth
Silver in air	74° 19′	43° 48′
" in water	71° 28′	44° 03′
" in turpentine	69° 16′	43° 21′
Gold in air	76° 0′	35° 27′
" in water	72° 46′	36° 23′
" in carbon disulphide	70° 03′	36° 48′

In a second series of experiments, Sir J. Conroy[3] found the following values.

Medium	Red		Yellow		Blue	
	P.I.	P.A.	P.I.	P.A.	P.I.	P.A.
Gold in air	73° 57′	41° 52′	71° 43′	41° 14′	67° 10′	35° 40′
do. water	70° 24′	42° 27′	67° 39′	41° 15′	63° 20′	36° 11′
do. carbon disulphide	69° 24′	42° 33′	66° 36′	41° 41′	60° 05′	36° 57′
Silver in air	76° 29′	43° 51′	74° 37′	43° 22′	71° 33′	43° 00′
do. water	73° 55′	44° 02′	72° 15′	44° 09′	67° 26′	43° 26′
do. carbon tetrachloride	72° 39′	44° 20′	71° 39′	43° 40′	66° 58′	44° 31′

The following table shows the percentage of light reflected at different angles of incidence from the following mirrors[4].

Angle of incidence	Silver	Steel	Tin	Speculum metal
10	70·05	54·38	39·76	66·13
20	70·06	55·39	40·28	66·88
30	71·35	54·93	44·38	66·87
40	70·87	55·62	44·11	67·26
50	72·49	56·74	47·48	67·26
60	74·19	57·63	50·60	66·32
65	73·58	58·37	52·32	66·53
70	74·63	58·09	54·97	67·65
75	77·25	58·69	58·85	67·43
80	81·19	63·56	65·08	70·17

[1] *Pogg. Ann.* Vol. CXXVIII. p. 541.
[2] *Proc. Roy. Soc.* Vol. XXVIII. p. 242; *Ibid.* pp. 248 and 250.
[3] *Proc. Roy. Soc.* Vol. XXXI. pp. 490, 496.
[4] *Proc. Roy. Soc.* Vol. XXXV. pp. 31, 32; and Vol. XXXVI. p. 187.

381. The ratio of the velocity of light in air to that in metals, has been investigated experimentally by Quincke, Wernicke, Voigt[1] and Kundt[2].

The values which Kundt has obtained for this ratio are given in the following table for red, white and blue light.

	Red	White	Blue
Silver		0·27	
Gold	0·38	0·58	1·00
Copper	0·45	0·65	0·95
Platinum	1·76	1·64	1·44
Iron	1·81	1·73	1·52
Nickel	2·17	2·01	1·85
Bismuth	2·61	2·26	2·13

From this table it appears, that the velocity of light in silver is nearly four times as great as in vacuo; but the dispersion was so small, that it could not be measured. Also in gold and copper, the velocity is greater than in vacuo, and the dispersion is normal; but in the other four metals it is anomalous.

Beer[3] has calculated the above ratio according to Cauchy's theory, from Jamin's observations on reflection. He found, that silver exhibited no marked dispersion, and that the mean ratio of the velocities was 0·25. Copper showed strong normal dispersion, and for the red rays the ratio was less than unity; iron, on the contrary, showed anomalous dispersion, giving $\mu_{\text{red}} = 2{\cdot}54$, $\mu_{\text{violet}} = 1{\cdot}47$, where μ is the ratio of the velocity of light in air to that in the metal.

382. Kundt also found, that there is a close relation between the velocity of light in metals, and their electrical conductivities. In the accompanying table, the velocity of light and the electrical conductivity of silver are both taken to be 100, and the conductivities are taken from Everett's *Units and Physical Constants*, p. 159.

[1] *Wied. Ann.* Vol. XXIII. pp. 104—147; Vol. XXV. pp. 95—114.

[2] *Sitz. der Kön. Preuss. Akad. der Wissen.*, 1888; translated *Phil. Mag.* July 1888.

[3] *Pogg. Ann.* Vol. XCII. p. 417.

Metal	Conductivity	Velocity of Light
Silver	100	100
Gold	71	71
Copper	94	60
Platinum	16·6	15·3
Iron	15·4	14·9
Nickel	12·0	12·4
Bismuth	1·1	10·3

With the exception of copper and bismuth, it appears that there is a fair agreement between the two sets of numbers.

383. Eisenlohr in the paper referred to in § 371, has applied Jamin's experimental results to calculate the quantities R and α by means of Cauchy's formulæ, and some of the values found by him are given in the following table.

Metal	Extreme Red		Yellow		Blue	
	α	log R	α	log R	α	log R
Copper	53° 37′	·4395	39° 45′	·3962	29° 45′	·3698
Silver	82° 46′	·5676	79° 31′	·4516	77° 58′	·3374
Speculum metal	57° 37′	·6111	51° 55′	·5078	51° 33′	·4605
Steel	31° 29′	·6621	32° 10′	·6102	36° 28′	·5782
Zinc	30° 04′	·5882	37° 38′	·5207	44° 05′	·4589

Now the value of μ^2 is

$$R^2(\cos 2\alpha + \iota \sin 2\alpha),$$

from which we see that for silver and speculum metal, μ^2 must be a complex quantity, whose *real part is negative*. For steel, the real part of μ^2 is positive; for copper it is negative for red light, and positive for yellow and blue; whilst for zinc it is positive for red, yellow and blue, but is negative for the remainder of the spectrum, since Eisenlohr found that for indigo $\alpha = 46° 23'$, and for the extreme violet $\alpha = 49° 08'$.

384. The circumstance that Cauchy's formulæ lead to the conclusion, that for certain metals the real part of μ^2 must be negative, has led to an important criticism by Lord Rayleigh[1], which we shall now consider.

If we suppose that the opacity of metals can be represented mathematically by a term proportional to the velocity, the equa-

[1] Hon. J. W. Strutt, *Phil. Mag.* May, 1872.

tion of motion within the metal, upon the elastic solid theory, may be written

$$\rho_1 \frac{d^2 w_1}{dt^2} + h \frac{dw_1}{dt} = n \left(\frac{d^2 w_1}{dx^2} + \frac{d^2 w_1}{dy^2} \right) \ldots\ldots\ldots\ldots (29),$$

where h is necessarily a positive constant.

The equation of motion outside the metal, will be

$$\rho \frac{d^2 w}{dt^2} = n \left(\frac{d^2 w}{dx^2} + \frac{d^2 w}{dy^2} \right).$$

To solve these equations assume

$$w = A \epsilon^{\frac{2\iota\pi}{V\tau}(-x\cos i + y \sin i - Vt)} + A' \epsilon^{\frac{2\iota\pi}{V\tau}(x\cos i + y \sin i - Vt)}$$

$$w_1 = A_1 \epsilon^{\frac{2\iota\pi}{V_1\tau}(-x\cos r + y \sin r - V_1 t)}.$$

Substituting in (29), we shall obtain

$$\mu^2 = \frac{\rho_1}{\rho} + \frac{\iota h \tau}{2\pi\rho}.$$

Under these circumstances, it follows that μ^2 is a complex quantity, whose real part is *positive;* hence α must lie between 0 and $\frac{1}{4}\pi$. Lord Rayleigh's investigation accordingly shows, that for silver and all metals for which $\alpha > \frac{1}{4}\pi$, reflection cannot be accounted for on the elastic solid theory, by the introduction of a viscous term.

385. When we consider the electromagnetic theory of light, it will be shown, that if we attempt to explain metallic reflection by taking into account the conductivity of the metal, we shall be led to equations of the same form. Hence metallic reflection cannot be completely explained, upon the electromagnetic theory, by means of this hypothesis.

386. We shall now show, that the circumstance of the square of the pseudo-refractive index being a complex quantity, whose real part is negative, may be explained by Von Helmholtz' theory.

Measuring the axis of z in the direction of propagation, and the axis of x in the direction of vibration, the equations of motion (11) and (10) of § 357 and 356, are

$$\rho \frac{d^2 u}{dt^2} = n \frac{d^2 u}{dz^2} + \alpha^2 (u_1 - u),$$

$$\rho_1 \frac{d^2 u_1}{dt^2} + 2h \frac{du_1}{dt} = \alpha^2 (u - u_1) - \beta^2 u_1.$$

Since we require a solution in which μ^2 is a complex quantity, we must not neglect the viscous term, and we shall find it convenient to conduct the integration of these equations, in a manner somewhat different from that of § 357.

Assume
$$u = A\epsilon^{2\iota\pi/\tau\,.\,(z/V-t)},$$
$$u_1 = A_1\epsilon^{2\iota\pi/\tau\,.\,(z/V-t)},$$

then
$$\left\{\frac{4\pi^2}{\tau^2}\left(\rho - \frac{n}{V^2}\right) - \alpha^2\right\}A = -\alpha^2 A_1,$$
$$\left\{4\pi^2\rho_1\left(\frac{1}{\tau^2} - \frac{1}{\kappa^2}\right) - \frac{h^2}{\rho_1} - \alpha^2 + \frac{4\iota\pi h}{\tau}\right\}A_1 = -\alpha^2 A,$$

whence
$$\left\{\frac{4\pi^2}{\tau^2}\left(\rho - \frac{n}{V^2}\right) - \alpha^2\right\}\left\{4\pi^2\rho_1\left(\frac{1}{\tau^2} - \frac{1}{\kappa^2}\right) - \frac{h^2}{\rho_1} - \alpha^2 + \frac{4\iota\pi h}{\tau}\right\} = \alpha^4 \ldots (30).$$

If ρ_0 be the density of the ether, and U the velocity of light in free space, $n/\rho_0 = U^2$; also since the pseudo-index of refraction of a metal is defined to be the ratio of U/V, we obtain from (30)

$$\mu^2 = \frac{\rho}{\rho_0} - \frac{\alpha^2\tau^2}{4\pi^2\rho_0}\left\{1 + \frac{\alpha^2\tau^2\kappa^2}{4\pi^2\rho_1(\kappa^2 - \tau^2) - (h^2/\rho_1 + \alpha^2)\kappa^2\tau^2 + 4\iota\pi h\kappa^2\tau}\right\} (31).$$

Rationalizing the denominator, we see that the imaginary part of μ^2 is positive, whilst the real part is equal to

$$\frac{\rho}{\rho_0} - \frac{\alpha^2\tau^2}{4\pi^2\rho_0}\left\{1 + \frac{\alpha^2\tau^2\kappa^2\{4\pi^2\rho_1(\kappa^2 - \tau^2) - (h^2/\rho_1 + \alpha^2)\kappa^2\tau^2\}}{\{4\pi^2\rho_1(\kappa^2 - \tau^2) - (h^2/\rho_1 + \alpha^2)\kappa^2\tau^2\}^2 + 16\pi^2h^2\kappa^4\tau^2}\right\} (32).$$

387. In order to apply this result to metallic reflection, we shall suppose that h is a small quantity, whose square may be neglected, under which circumstances, the real part of μ^2 which we shall denote by ν^2, becomes equal to

$$\nu^2 = \frac{\rho}{\rho_0} - \frac{\alpha^2\tau^2}{4\pi^2\rho_0}\left\{1 + \frac{\alpha^2\kappa^2\tau^2}{4\pi^2\rho_1(\kappa^2 - \tau^2) - \alpha^2\kappa^2\tau^2}\right\} \ldots\ldots\ldots (33).$$

This expression is the same as the square of the refractive index of a substance, which produces anomalous dispersion, and has a single absorption band; and it follows from § 359, that it may be negative in two distinct ways.

In the first place, if τ_1 is the least value of τ for which $\nu^2 = 0$, and τ_2 is the value of τ, which makes the denominator of the third term zero, the real part of μ^2 will be negative for values of τ lying between τ_1 and τ_2. In the second place, there is another value τ_3 of τ, which is greater than τ_2, for which $\nu^2 = 0$; and for all values of $\tau > \tau_3$, ν^2 is negative. We may therefore explain

reflection from silver in two distinct ways. In the first place we may suppose, that the period of the free vibrations is such, that throughout the luminous portion of the spectrum, and some distance beyond it and on either side, τ lies between τ_1 and τ_2; or in the second place we may suppose, that throughout this range $\tau > \tau_3$. Now metals reflect rays of dark heat[1] in much the same way as they reflect light; accordingly if we adopted the first hypothesis, it would be necessary to suppose, that κ the free period of the matter vibrations, lies below the infra-red portion of the spectrum; if on the other hand, we adopted the second hypothesis, it would be necessary to suppose, that κ corresponds to a point in or above the ultra-violet portion of the spectrum. To explain reflection from steel, we must suppose that κ is such, that throughout the luminous portion of the spectrum and some distance beyond, τ is less than τ_1, or lies between τ_2 and τ_3. To explain reflection from copper, we must suppose that either τ_1 or τ_3 corresponds to a point of the spectrum intermediate between the red and yellow, since in going up the spectrum, the real part of μ^2 passes through zero, from a negative to a positive value. But in the case of zinc, the real part of μ^2 begins by being positive, and then passes through zero to a negative value at a point between the blue and violet. The theory in its present form is therefore not applicable to zinc. It is however necessary to point out, that the theory which has been developed, only applies to a medium having a single absorption band; whereas there is no *a priori* reason why metals should not possess several. A theory such as von Helmholtz' could be extended, so as to apply to a medium having a number of absorption bands; and there can be little doubt, that the real part of μ^2 would be given by an expression of much the same form, as that furnished by Lord Kelvin's theory.

388. The investigations of the last two Chapters, will give the reader some idea of the various theories relating to the mutual reaction between ether and matter, which have been proposed to explain dispersion and metallic reflection. Further information upon this subject, will be found in Glazebrook's *Report on Optical Theories*[2], where a variety of theories due to Lommel, Voigt, Ketteler and others are considered. It must however be con-

[1] Magnus, *Pogg. Ann.* Vol. CXXXIX.
[2] *Brit. Assoc. Rep.* 1886.

fessed, that most of these theories are of a somewhat tentative and unsatisfactory character; and depend to a great extent upon unproved hypotheses and assumptions made during the progress of the work, for the purpose of obtaining certain analytical results. The fundamental hypothesis, first suggested by Stokes[1], and afterwards more fully developed by Sellmeier[2], that these phenomena are due to the fact, that some of the free periods of the vibrations of the molecules of matter fall within the limits of the periods of the visible spectrum, is deserving of attentive consideration and development. This hypothesis is quite independent of any suppositions, which may be made respecting the physical constitution of the ether; since any medium, which is capable of propagating waves, would produce vibrations of the molecules of the matter embedded in it, of the same kind as those we have been discussing.

389. We shall see in the next Chapter, that the electromagnetic theory of light presupposes the existence of a medium or ether; and that the general equations of the electromagnetic field show, that the motion of this medium is governed by equations, which are nearly identical with those furnished by the elastic solid theory. When electromagnetic waves impinge upon the molecules of a material substance, the latter are thrown into a state of vibration, and by making additional assumptions respecting the mutual reaction of ether and matter, we may translate many of the investigations based upon the elastic solid theory, into the language of the electromagnetic theory. Moreover most transparent bodies are dielectrics, whilst metals are conductors of electricity; and certain metals such as iron, cobalt and nickel are strongly magnetic. We should therefore be led to expect, that there would be a marked difference between the propagation of electromagnetic waves in dielectrics on the one hand, and in metals on the other hand. Unfortunately the electromagnetic theory, in the form in which it has hitherto been developed, does not readily lend itself to an explanation of dispersion and metallic reflection; and it must be admitted these phenomena have not as yet been satisfactorily accounted for.

[1] *Phil. Mag.*, March 1860, p. 196.

[2] *Pogg. Ann.* Vols. CXLII. p. 272; CXLV. pp 399, 520; CXLVII. pp. 386, 525.

CHAPTER XIX.

THE ELECTROMAGNETIC THEORY.

390. THE electromagnetic theory of light, which was first proposed by the late Prof. Clerk-Maxwell, supposes that the sensation of light is produced by means of an electromagnetic disturbance, which is propagated in a medium; and we cannot do better than to give the fundamental idea of this theory in Maxwell's own words[1]:—

"To fill all space with a new medium, whenever any new phenomenon is to be explained, is by no means philosophical, but if the study of two different branches of science has independently suggested the idea of a medium, and if the properties which must be attributed to the medium in order to account for electromagnetic phenomena, are of the same kind as those which we attribute to the luminiferous medium in order to account for the phenomena of light, the evidence of the physical existence of the medium will be considerably strengthened.

"But the properties of bodies are capable of quantitative measurement. We therefore obtain the numerical value of some property of the medium, such as the velocity with which a disturbance is propagated through it, which can be calculated from electromagnetic experiments, and observed directly in the case of light. If it should be found that the velocity of propagation of electromagnetic disturbances is the same as the velocity of light, and this not only in air, but in other transparent media, we shall have strong reasons for believing that light is an electromagnetic phenomenon, and that a combination of the

[1] *Electricity and Magnetism*, Vol. II. p. 383.

optical with the electrical evidence will produce a conviction of the reality of the medium, similar to that which we obtain, in the case of other kinds of matter, from the combined evidence of the senses."

391. We shall now proceed to apply the general equations of the electromagnetic field, to obtain the velocity of propagation of an electromagnetic disturbance.

The equations of electromotive force are[1]

$$\left.\begin{aligned} P &= -\frac{dF}{dt} - \frac{d\psi}{dx} \\ Q &= -\frac{dG}{dt} - \frac{d\psi}{dy} \\ R &= -\frac{dH}{dt} - \frac{d\psi}{dz} \end{aligned}\right\} \qquad (1).$$

The equations of magnetic induction are,

$$\left.\begin{aligned} a &= \frac{dH}{dy} - \frac{dG}{dz} \\ b &= \frac{dF}{dz} - \frac{dH}{dx} \\ c &= \frac{dG}{dx} - \frac{dF}{dy} \end{aligned}\right\} \qquad (2).$$

The equations of the currents are,

$$\left.\begin{aligned} 4\pi u &= \frac{d\gamma}{dy} - \frac{d\beta}{dz} \\ 4\pi v &= \frac{d\alpha}{dz} - \frac{d\gamma}{dx} \\ 4\pi w &= \frac{d\beta}{dx} - \frac{d\alpha}{dy} \end{aligned}\right\} \qquad (3),$$

Maxwell's notation being employed.

392. If the medium is magnetically isotropic, the magnetic force and the magnetic induction will be connected together by the equations

$$a = \mu\alpha, \quad b = \mu\beta, \quad c = \mu\gamma \qquad (4),$$

where μ is the magnetic permeability of the medium.

[1] *Electricity and Magnetism*, Vol. II. Chapter IX.

393. If the medium were electrostatically isotropic, the electromotive force in any direction, would be proportional to the electric displacement in the same direction; but if the medium is æolotropic, the relation between electromotive force and electric displacement will depend upon the peculiar constitution of the medium. We have already pointed out, that all doubly-refracting media possess three rectangular planes of symmetry; and we shall now show, that double refraction can be explained by supposing, that the medium is electrostatically æolotropic.

If the axes of symmetry are the axes of coordinates, the equations connecting the electromotive force and electric displacement may be written

$$P = 4\pi f/K_1, \quad Q = 4\pi g/K_2, \quad R = 4\pi h/K_3 \;\ldots\ldots\ldots (5),$$

where K_1, K_2, K_3 are the three principal electrostatic capacities.

If the medium were a conductor, the equations between the electromotive force and the conduction current would be

$$p = C_1 P, \quad q = C_2 Q, \quad r = C_3 R \;\ldots\ldots\ldots\ldots\ldots (6),$$

where C_1, C_2, C_3 are the three principal conductivities. If we suppose the medium isotropic as regards conduction, the three C's will be equal.

The equations connecting the true current, with the electric displacement and conduction current, are

$$u = \dot{f} + p, \quad v = \dot{g} + q, \quad w = \dot{h} + r \;\ldots\ldots\ldots\ldots\ldots (7).$$

Since most transparent media are good insulators, we shall suppose that the conduction current is zero, which requires that $C_1 = C_2 = C_3 = 0$.

394. We can now obtain the equations of electric displacement.

From (1) and (2) we obtain

$$\left.\begin{aligned} -\frac{da}{dt} &= \frac{dR}{dy} - \frac{dQ}{dz} \\ -\frac{db}{dt} &= \frac{dP}{dz} - \frac{dR}{dx} \\ -\frac{dc}{dt} &= \frac{dQ}{dx} - \frac{dP}{dy} \end{aligned}\right\} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8).$$

From (3) and (7), we obtain

$$4\pi\mu\dot{u} = 4\pi\mu\ddot{f} = \frac{d\dot{c}}{dy} - \frac{d\dot{b}}{dz} \quad \text{.................. (9)},$$

with two similar equations. Eliminating $\dot{a}$, $\dot{b}$, $\dot{c}$ from (9) by means of (8) and putting

$$\mathfrak{P} = \frac{dP}{dx} + \frac{dQ}{dy} + \frac{dR}{dz} \quad \text{.................. (10)},$$

we shall obtain

$$\left.\begin{aligned} 4\pi\mu\frac{d^2f}{dt^2} &= \nabla^2 P - \frac{d\mathfrak{P}}{dx} \\ 4\pi\mu\frac{d^2g}{dt^2} &= \nabla^2 Q - \frac{d\mathfrak{P}}{dy} \\ 4\pi\mu\frac{d^2h}{dt^2} &= \nabla^2 R - \frac{d\mathfrak{P}}{dz} \end{aligned}\right\} \quad \text{..................(11)}.$$

In proving these equations, we have not as yet made any assumption respecting the relation between electromotive force and electric displacement. Let us now substitute the values of P, Q, R from (5), also let

$$\mu K_1 = A^{-2}, \quad \mu K_2 = B^{-2}, \quad \mu K_3 = C^{-2} \quad \text{......... (12)},$$

$$\Omega = A^2\frac{df}{dx} + B^2\frac{dg}{dy} + C^2\frac{dh}{dz} \quad \text{.............. (13)},$$

then (11) become

$$\left.\begin{aligned} \frac{d^2f}{dt^2} &= A^2\nabla^2 f - \frac{d\Omega}{dx} \\ \frac{d^2g}{dt^2} &= B^2\nabla^2 g - \frac{d\Omega}{dy} \\ \frac{d^2h}{dt^2} &= C^2\nabla^2 h - \frac{d\Omega}{dz} \end{aligned}\right\} \quad \text{.................... (14)}.$$

These are the equations which are satisfied by the electric displacement.

395. To find the equations of magnetic induction, differentiate the first of (8) with respect to t, and then substitute the values of $\dot{R}$, $\dot{Q}$ from (5), and we shall obtain

$$-\frac{d^2a}{dt^2} = 4\pi\left(\frac{1}{K_3}\frac{d\dot{h}}{dy} - \frac{1}{K_2}\frac{d\dot{g}}{dz}\right).$$

Substituting the values of $\dot{h}$, $\dot{g}$ from (9), taking account of (12),

and of the other two equations, which can be written down by symmetry, we shall finally obtain

$$\left.\begin{aligned}\frac{d^2a}{dt^2} &= B^2\frac{d}{dz}\left(\frac{da}{dz}-\frac{dc}{dx}\right)-C^2\frac{d}{dy}\left(\frac{db}{dx}-\frac{da}{dy}\right)\\ \frac{d^2b}{dt^2} &= C^2\frac{d}{dx}\left(\frac{db}{dx}-\frac{da}{dy}\right)-A^2\frac{d}{dz}\left(\frac{dc}{dy}-\frac{db}{dz}\right)\\ \frac{d^2c}{dt^2} &= A^2\frac{d}{dy}\left(\frac{dc}{dy}-\frac{db}{dz}\right)-B^2\frac{d}{dx}\left(\frac{da}{dz}-\frac{dc}{dx}\right)\end{aligned}\right\}\text{......} \ (15).$$

396. With the exception of the terms in δ, equations (15) are of the same form, as equations (6) of § 247, which determine the displacements of the medium on Green's Theory: whilst equations (14) are identical with (7) of § 247, which determine the rotations on Green's theory.

It therefore follows, that the velocities of propagation both of the electric displacement and the magnetic induction are determined by Fresnel's construction, and that the direction of the electric displacement is in the front of the wave, and is perpendicular to the plane of polarization; whilst the magnetic displacement is in the front of the wave and is perpendicular to the electric displacement. If therefore we adopt the hypothesis, that the direction of the disturbance which constitutes light is perpendicular to the plane of polarization, we must suppose that this disturbance is represented by the *electric displacement.*

397. On account of the importance of the subject, it will be desirable to deduce these results directly from the equations. Let S be the resultant electric displacement; λ, μ, ν its direction cosines; also let l, m, n be those of the normal to the wave, V the velocity of propagation of the latter. Then we may write

$$S = F(lx+my+nz-Vt) \text{ .... } (16),$$

accordingly $$f=S\lambda,\quad g=S\mu,\quad h=S\nu \text{ .... } (17).$$

Since $$\frac{df}{dx}+\frac{dg}{dy}+\frac{dh}{dz}=0,$$

it follows that $$l\lambda+m\mu+n\nu=0 \text{ .... } (18),$$

which shows that the electric displacement lies in the front of the wave.

Substituting the values of f, g, h from (17) in (14), we obtain

$$\left.\begin{aligned} \lambda(V^2-A^2)+\ l(A^2l\lambda+B^2m\mu+C^2n\nu)=0 \\ \mu(V^2-B^2)+m(A^2l\lambda+B^2m\mu+C^2n\nu)=0 \\ \nu(V^2-C^2)+\ n(A^2l\lambda+B^2m\mu+C^2n\nu)=0 \end{aligned}\right\} \ldots\ldots (19),$$

from which equations combined with (18), we at once obtain

$$\frac{l^2}{V^2-A^2}+\frac{m^2}{V^2-B^2}+\frac{n^2}{V^2-C^2}=0 \ldots\ldots\ldots\ldots (20).$$

This equation shows, that the electric displacement is propagated in a doubly-refracting medium according to Fresnel's law.

Multiplying (19) by λ, μ, ν and adding, we obtain

$$V^2=A^2\lambda^2+B^2\mu^2+C^2\nu^2 \ldots\ldots\ldots\ldots\ldots\ldots (21).$$

Also from the same equations it can be shown, that

$$\frac{l}{\lambda}(B^2-C^2)+\frac{m}{\mu}(C^2-A^2)+\frac{n}{\nu}(A^2-B^2)=0 \ldots\ldots (22),$$

and $$(V^2-A^2)\lambda/l=(V^2-B^2)\mu/m=(V^2-C^2)\nu/n \ldots\ldots (23).$$

Let P be the point of contact of the tangent plane to the wave surface, and let l, m, n be its direction cosines; also let Y be the foot of the perpendicular from the origin on to this tangent plane. Then if L, M, N be the direction cosines of PY, it is known from the geometry of the wave surface, see (19) of § 109 and § 112, that

$$(V^2-A^2)L/l=(V^2-B^2)M/m=(V^2-C^2)N/n,$$

accordingly by (23) we have

$$L/\lambda=M/\mu=N/\nu;$$

which shows that the electric displacement is perpendicular to the plane of polarization.

By treating equations (15) in the same way as (14), it can be shown that the magnetic induction lies in the plane of the wave-front, and that it is propagated at the same rate as the electric displacement.

398. We must now determine the magnetic force in terms of the electric displacement.

Substituting the values P, Q, R from (5) in (8), and taking account of (12), we obtain

$$\frac{da}{dt}=4\pi\left(B^2\frac{dg}{dz}-C^2\frac{dh}{dy}\right).$$

But $$\frac{dg}{dz} = \mu\frac{dF}{dz} = -\frac{n\mu}{V}\frac{dF}{dt} = -n\mu\dot{S}/V,$$

whence
$$\left.\begin{aligned} \alpha &= 4\pi\,(C^2m\nu - B^2n\mu\,)\,S/V \\ \beta &= 4\pi\,(A^2n\lambda - C^2l\nu\;\;)\,S/V \\ \gamma &= 4\pi\,(B^2l\mu\;\; - A^2m\lambda)S/V \end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots(24).$$

Multiplying these equations by l, m, n and adding, we obtain
$$l\alpha + m\beta + n\gamma = 0,$$
which shows that the magnetic force, and therefore the magnetic induction, lies in the plane of the wave-front.

Multiplying by λ, μ, ν and taking account of (22) we obtain
$$\lambda\alpha + \mu\beta + \nu\gamma = 0,$$
which shows that the magnetic force is perpendicular to the electric displacement.

Let λ', μ', ν' be the direction cosines of the magnetic force, then
$$l = \mu\nu' - \mu'\nu, \quad m = \nu\lambda' - \nu'\lambda, \quad n = \lambda\mu' - \lambda'\mu.$$

Substituting in the first of (24), and taking account of (21), we obtain
$$\alpha = 4\pi\,\{V^2\lambda' - \lambda\,(A^2\lambda\lambda' + B^2\mu\mu' + C^2\nu\nu')\}\;S/V;$$
but $\lambda' = m\nu - n\mu$ &c.; whence it follows from (22) that
$$A^2\lambda\lambda' + B^2\mu\mu' + C^2\nu\nu' = 0,$$
accordingly $$\alpha = 4\pi VS\lambda' \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (25),$$
and therefore the magnetic force, corresponding to an electric displacement S, is equal to $4\pi VS$.

In (24) put $l = 1$, $m = n = 0$; $\mu = 1$, $\lambda = \nu = 0$; then it follows, that when a wave is propagated along the positive direction of the axis of x, and the electric displacement is parallel to the axis of y, the magnetic force will be parallel to the axis of z.

399. We shall now find the direction of the electromotive force.

From (5) and (12) we have
$$P = 4\pi\mu_1A^2f, \quad Q = 4\pi\mu_1B^2g, \quad R = 4\pi\mu_1C^2h,$$
where μ_1 temporarily denotes the magnetic permeability, to distinguish it from μ the y-direction cosine of the electric displacement. Let χ be the angle between the radius vector and the normal to the tangent plane at its extremity, then
$$\cos\chi = V/r, \quad \cos(\tfrac{1}{2}\pi + \chi) = -\sin\chi = (\lambda x + \mu y + \nu z)/r \ldots (26),$$

where x, y, z are the coordinates of the point of contact. Now by the geometry of the wave surface

$$(r^2 - A^2)\lambda/x = (r^2 - B^2)\mu/y = (r^2 - C^2)\nu/z = -\frac{r^2 - V^2}{r \sin\chi} = -r\sin\chi,$$

by (21) and (26); whence

$$(r^2 - A^2)\lambda/Vx = (r^2 - B^2)\mu/Vy = (r^2 - C^2)\nu/Vz = -\tan\chi.$$

But $$x = \frac{lV(r^2 - A^2)}{V^2 - A^2}; \text{ \&c. \&c.};$$

whence

$$(V^2 - A^2)\lambda/l = (V^2 - B^2)\mu/m = (V^2 - C^2)\nu/n = -V^2\tan\chi,$$

and therefore by (19)

$$A^2 l\lambda + B^2 m\mu + C^2 n\nu = V^2 \tan\chi.$$

The component of the electromotive force along the wave normal is therefore

$$Pl + Qm + Rn = 4\pi\mu_1 (A^2 l\lambda + B^2 m\mu + C^2 n\nu) S = 4\pi\mu_1 V^2 S \tan\chi \quad \text{...........(27)},$$

and the component along the direction of displacement is

$$P\lambda + Q\mu + R\nu = 4\pi\mu_1 V^2 S \quad \text{.............. (28)}.$$

From (27) and (28) we see, that the resultant electromotive force is equal to $4\pi\mu_1 V^2 S \sec\chi$, and that its direction is perpendicular to the ray.

400. The preceding analysis contains a complete investigation of the propagation of electric and magnetic disturbances in an electrostatically æolotropic medium; and we have shown, that both disturbances are propagated with a velocity, which satisfies the same mathematical conditions as the velocity of propagation of light in a biaxal crystal, according to Fresnel's theory; but that the electric disturbance is perpendicular to what in optical language is known as the plane of polarization, whilst the magnetic disturbance lies in the plane of polarization. If therefore the sensation of light is the result of electromagnetic waves, we must conclude, that light is the effect of the electric displacement and not of the magnetic displacement (or magnetic induction). Before however we can decide whether light is an electromagnetic phenomenon, it is necessary to ascertain whether the electric and magnetic disturbances are propagated with a velocity, which is equal to or comparable with that of light.

401. In an isotropic medium, $K_1 = K_2 = K_3$, and Ω is zero; whence (14) become

$$\ddot{f} = (\mu K)^{-1}\nabla^2 f, \quad \ddot{g} = (\mu K)^{-1}\nabla^2 g, \quad \ddot{h} = (\mu K)^{-1}\nabla^2 h \ldots (29).$$

From these equations we see, that the velocity of propagation V is equal to $(\mu K)^{-\frac{1}{2}}$. If the medium is air, and we adopt the electrostatic system of units, $K = 1$, and $\mu = v^{-2}$, where v is the number of electrostatic units in one electromagnetic unit, whence $V = v$; or the velocity of propagation of light is equal to the number of electrostatic units in one electromagnetic unit. If on the other hand we adopt the electromagnetic system, $K = v^{-2}$, and $\mu = 1$, so that the equation $V = v$ is still true.

402. The methods of determining v are explained in Maxwell's *Electricity and Magnetism*, Vol. II. Ch. XIX., and are quite independent of the methods for determining the velocity of light; hence the agreement or disagreement of the values of V and v furnishes a test of the electromagnetic theory of light.

The following table, taken from Maxwell, gives the values of V and v in C. G. S. units.

Velocity of Light.		*Ratio of Electric Units.*	
Fizeau	31400×10^6	Weber	31074×10^6
Aberration &c., and Sun's parallax	30800×10^6	Maxwell	28800×10^6
Foucault	29836×10^6	Lord Kelvin	28200×10^6

From these results we see, that the velocity of light, and the ratio of units are quantities of the same order; but none of them can be considered to be determined with such a degree of accuracy as to enable us to assert, that one is greater than the other[1].

403. In all transparent media the magnetic permeability is very nearly equal to that of air, hence refraction must depend principally upon differences of specific inductive capacity. According to the electromagnetic theory, the dielectric capacity of a transparent medium is equal to the square of its index of refraction. But the index of refraction of light is different for different colours, being greater for light of short period; we must therefore select the index of refraction, which corresponds to waves of longest period, since these are the only waves whose motion can

[1] See also the note, p. 379.

be compared to the slow processes, by which the capacity of a dielectric can be determined.

The square root of the value of K for paraffin[1] is 1·405; whilst the index of refraction for waves of infinite period is about 1·422.

404. In discussing these experimental results Maxwell concludes as follows:—"The difference between these numbers is greater than can be accounted for by errors of observation, and shows that our theories of the structure of bodies must be much improved, before we can deduce their optical from their electrical properties. At the same time, I think, that the agreement between the numbers is such, that if no greater discrepancy were found between the numbers derived from the optical and the electrical properties of a considerable number of substances, we should be warranted in concluding that the square root of K, although it may not be the complete expression for the index of refraction, is at least the most important term in it."

405. In 1873, which was the date of publication of the first edition of Maxwell's treatise on *Electricity and Magnetism*, paraffin was the only transparent dielectric, whose electrostatic capacity had been determined. Since that date, the capacity of a variety of other media have been determined, and it has been found that for many substances, the square of the refractive index differs considerably from the value of the electrostatic capacity.

The experiments of Hopkinson[2] give the following results for the electrostatic capacity of Chance's glasses.

	ρ	K	K/ρ	μ
Light flint	3·2	6·85	2·14	1·574
Double extra-dense	4·5	10·1	2·25	1·710
Dense flint	3·66	7·4	2·02	1·622
Very light flint	2·87	6·57	2·29	1·541

In this table, ρ is the density, K is the electrostatic capacity, and μ is the index of refraction of the double line D of the spectrum.

[1] Gibson and Barclay, *Phil. Trans.* 1871, p. 573.
[2] *Phil. Trans.* 1878, p. 17.

A further series of experiments was made by Hopkinson[1], which gave the following results.

	ρ	K
Double extra-dense flint glass	4·5	9·896
Dense flint	3·66	7·376
Light flint	3·2	6·72
Very light flint	2·87	6·61
Hard crown	2·485	6·96
Plate glass	—	8·45
Paraffin	—	2·29

406. In the last paper an account is given of experiments made upon certain liquids; and the results are shown in the following table. The value of μ^2_∞ is calculated by means of Cauchy's formula

$$\mu^2 = \mu^2_\infty + b/\lambda^2.$$

	μ^2_∞	K
Petroleum spirit (Field's)	1·922	1·92
do. oil (Field's)	2·075	2·07
do. common	2·078	2·10
Ozokerit lubricating oil (Field's)	2·086	2·13
Turpentine (commercial)	2·128	2·23
Castor oil	2·153	4·78
Sperm oil	2·135	3·02
Olive oil	2·131	3·16
Neats' foot oil	2·125	3·07

From these tables it appears, that the vegetable and animal oils do not agree with Maxwell's theory, but the hydrocarbon oils do. But in the electrical experiments, the determination was effected by the charge and discharge of a condenser; and it must be recollected, that even when the time of charge and discharge is only 5×10^{-5} of a second, this period is many million times longer than the period of the waves of any portion of the visible spectrum.

407. The capacities of Iceland spar, fluor spar and quartz have been examined by Romich and Nowak[2], and give results which

[1] *Phil. Trans.* 1881, p. 355.
[2] *Wiener. Sitzb.* vol. LXX. part ii. p. 380.

are much in excess of the square of the refractive index. On the other hand, the same observers, and also Boltzmann, obtain for crystallized sulphur, a value of the capacity in reasonable accord with theory.

The experimental determinations of electrostatic capacities, made by Boltzmann for paraffin, colophonium and sulphur[1], and also for various gases[2]; by Silow for turpentine and petroleum[3]; by Schiller[4] and Wüllner[5] for plate glass, will be found in the papers referred to below.

Hertz's Experiments[6].

408. The rapidity of the propagation of electrical effects across space or any insulating medium, which has until recently eluded all attempts at measurement, early suggested to natural philosophers, that it might be connected with the mode of propagation of light across space. For all kinds of mechanical tremors in the matter of bodies are propagated comparatively slowly, in the manner of sound waves; while the propagation of free gravitation was shown long ago by Laplace, to be extremely rapid, even compared with that of light itself.

This suggested connection was enormously strengthened when Maxwell, who was the first to try to express the known equations of electrodynamic action in a form, which suggested and implied propagation across a medium, found that his system gave rise to electric waves of the same transverse character as waves of light, whose velocity of propagation is an electric constant, which on measurement turns out to be for a vacuum the same as the velocity of light. Nor is the fact that, except for media of simple and homogeneous chemical constitution, this agreement in velocities is not very generally observed, a serious drawback to the theory, when we consider the great difficulty of unravelling the complex effect of the molecules of matter on the propagation of light, and on the character of electric actions.

[1] *Pogg. Ann.* (1874), vol. CLI. pp. 482 and 531; vol. CLIII. p. 525.
[2] *Ibid.* (1875), vol. CLV. p. 403.
[3] *Ibid.* (1875), vol. CLVI. p. 389; (1876), vol. CLVIII. p. 306.
[4] *Ibid.* (1874), vol. CLII. p. 535.
[5] *Ibid. New Series*, vol. I. pp. 247 and 361.
[6] I am indebted to Mr Larmor for §§ 408—409.

409. There remained however another side of the subject to explore, in the detection and systematic examination of actual electric vibrations propagated across space. The difficulties in the way were (i) to obtain a vibrating electric system, with periods high enough to give waves of manageable length; (ii) to obtain some method of detecting their propagation. These difficulties have been successfully surmounted within the last few years by Hertz[1]. The vibrations were set up by the snap of an electric discharge between two conductors, whose capacity and self-induction were so arranged as to give a wave-length of the order of magnitude of ordinary waves of sound, or even down to a few inches. The detector in one form consists of a wire circuit, with a minute spark-gap in it. When placed in a field across which waves are travelling, whose period is the same as that of the free electric oscillations of the circuit itself, the latter acts as a resonator, and reveals the presence of the waves by sparking. It was found by Hertz, that such a resonator was excited at equidistant positions in front of the vibrator, corresponding to half a wave-length; and that the circumstances corresponded in all respects to the mode of propagation of the transverse electric waves of Maxwell's theory. It is now pretty certain, that the radiation from the vibrator contains a wide spectrum of wave-lengths. The vibrator being worked by a rapid torrent of sparks from an induction coil, each spark sets up an electric vibration swaying in it, which is very rapidly damped by radiation, even in a very few swings. The succession of sparks thus sends out a succession of disturbances, which have no single definite period, but are capable of being decomposed in Fourier's manner into a whole spectrum of simple waves travelling out into the medium. Of these the resonator takes up the appropriate one, and reinforces it; thus the observed wave-length corresponds to the period of the resonator, and is in fact different for different resonators. This mode of explanation appears to require, that when an electric vibration is started in a resonator, it persists sensibly over the period between two successive sparks of the primary; and therefore that the resonator should present a small surface for radiation.

At any rate, Hertz's experiments have firmly established that electric radiation does exist; and that its properties are exactly on the lines indicated by the appropriate *a priori* electric theories.

[1] *Wied. Ann.* vols. XXXI. to XXXVI.

Thus we can experiment with electrical waves of sensible length, and thereby check theoretical developments; and we can push on the correspondence in properties between such waves and the waves of light, which are of very minute length. And it hardly admits of doubt, that in the case of a vacuum, where the complication of ponderable molecules with their disturbing free periods does not come in, absolute continuity will be found to exist in the transition from the one class to the other. But in the case of ponderable media, the two classes of waves will be influenced by free molecular periods of wholly different orders; so that any minute numerical correspondence is perhaps not to be anticipated.

410. By means of his experiments, Hertz proved the interference, reflection and polarization of electromagnetic waves; and from certain calculations based upon the results of his experiments, he has shown that the velocity of electromagnetic waves is approximately the same as that of light. Trouton[1] has further proved experimentally, that if electromagnetic waves are incident at the polarizing angle upon a bad conductor, the waves are not reflected when the direction of magnetic force is perpendicular to the plane of incidence; but when the direction of the former is parallel to the latter, reflection takes place at all angles of incidence. This experiment confirms Fresnel's hypothesis, that the vibrations of polarized light are perpendicular to the plane of polarization; and that upon Maxwell's theory, the disturbance which gives rise to optical effects is represented by the electric displacement.

A complete discussion of the experiments of Hertz, and of the various other theories on the connection between light and electricity, belongs rather to a treatise on electromagnetism than to one on light. The reader, who desires further information upon these matters, is recommended to consult the original memoirs, and also Poincaré's *Électricité et Optique*, Part II., in which a very full account of Hertz's experiments is given.

[1] *Nature*, 22nd Aug. 1889; see also Fitzgerald, *Proc. Roy. Inst.* March 21st, 1890.

Intensity of Light.

411. The intensity of light is usually measured on the electromagnetic theory, by the average energy per unit of volume.

In a doubly refracting medium, the electrostatic energy per unit of volume is

$$\tfrac{1}{2}(Pf + Qg + Rh) = 2\pi\mu_1(A^2\lambda^2 + B^2\mu^2 + C^2\nu^2)S^2$$
$$= 2\pi\mu_1 V^2S^2,$$

by (21).

The electrokinetic energy is

$$\frac{1}{8\pi}\mu_1(\alpha^2 + \beta^2 + \gamma^2) = 2\pi\mu_1 V^2S^2,$$

by (25).

It therefore follows that in any medium, the electrostatic and electrokinetic energies are equal.

Let E be the total energy, and let

$$S = \mathfrak{A}\cos\frac{2\pi}{\lambda}(lx + my + nz - Vt),$$

then $$E = 2\pi\mu_1\mathfrak{A}^2V^2\left\{1 + \cos\frac{4\pi}{\lambda}(lx + my + nz - Vt)\right\}.$$

The energy therefore consists of two parts, one of which is a constant term, and the other is a periodic term. The first term is the average energy per unit of volume; and consequently the intensity of light on the electromagnetic theory, is proportional to the product of the magnetic permeability of the medium, the square of the amplitude, and the square of the velocity of propagation in the direction in which the wave is travelling.

Conditions to be satisfied at the Surface of Separation of Two Media.

412. The conditions of continuity of force require, that the electric and magnetic forces *parallel* to the surface of separation should be the same in both media. These conditions furnish four equations.

As regards the conditions to be satisfied perpendicular to the surface of separation, Maxwell has shown, Vol. I. § 83, that if P, P' be the normal components of the electromotive force at the surface

of separation of two media, whose specific inductive capacities are K, K', then

$$PK - P'K' = 0;$$

whence the components of the electric displacement perpendicular to the surface of separation must be the same in both media.

Again, if μ, μ' be the magnetic permeabilities of the two media, and α, α' the normal components of the magnetic force, Maxwell has shown Vol. II. § 428, that

$$\mu\alpha - \mu'\alpha' = 0;$$

whence the components of the magnetic induction perpendicular to the surface of separation must be the same in both media[1].

413. These conditions furnish altogether six equations, but we shall presently show that they reduce to only four; inasmuch as it will be proved later on, that the condition, that the electric displacement perpendicular to the surface of separation should be continuous, is analytically equivalent to the condition, that the magnetic force parallel to the line of intersection of the wave-front with the surface of separation should be the same in both media; and that the condition, that the magnetic induction perpendicular to the surface of separation should be continuous, is analytically equivalent to the condition, that the electric force parallel to the line of intersection of the wave-front with the surface of separation should be the same in both media.

414. The equations of motion (29) of an isotropic medium are of the same form, as those furnished by the elastic solid theory when δ is absolutely zero; for since there is no accumulation of free electricity $df/dx + dg/dy + dh/dz$ is always zero. We may therefore explain a variety of optical phenomena relating to isotropic media, by means of the electromagnetic theory just as well as by the elastic solid theory. There is however one important distinction between the two theories, viz. that the supposition that $\delta = 0$, requires that $m = \infty$, and accordingly $m\delta$ may be finite; and in studying Green's theory of the reflection and refraction of light, we saw that under these circumstances it was necessary to

[1] The continuity of electric displacement and magnetic induction can be at once deduced from the condition, that these quantities both satisfy an equation of the same form as the equation of continuity of an incompressible fluid. This equation is likewise the condition, that the directions of these quantities should be parallel to the wave-front.

introduce a pressural wave. Nothing of the kind occurs in the electromagnetic theory, and accordingly we are relieved from one of the difficulties of the elastic solid theory. We shall now proceed to consider the reflection and refraction of light at the common surface of two isotropic media.

Reflection and Refraction[1].

415. Instead of beginning with the case of an isotropic medium, we shall suppose that the reflecting surface consists of a plate of Iceland spar, which is cut perpendicularly to its axis; so that we can pass to the case of an isotropic medium by putting

$$a = c.$$

The wave surface consists of the sphere

$$x^2 + y^2 + z^2 = c^2,$$

and the planetary ellipsoid

$$x^2/c^2 + (y^2 + z^2)/a^2 = 1.$$

416. Let A, A', A_1 be the amplitudes of the incident, reflected and refracted waves; and let us first suppose that the incident light is polarized in the plane of incidence, so that the refracted ray is the ordinary ray.

The condition that the electric forces parallel to the plane of incidence should be continuous, gives

$$V^2(A + A') = c^2 A_1 \qquad\qquad (30).$$

The condition that the corresponding components of magnetic induction should be continuous, gives

$$(A - A')\, V \cos i = A_1 c \cos r \qquad\qquad (31).$$

But if I, I', I_1 be the square roots of the intensities, it follows from § 411, that

$$\frac{I}{AV} = \frac{I'}{A'V} = \frac{I_1}{A_1 c},$$

whence (30) and (31) become

$$(I + I') \sin i = I_1 \sin r,$$

$$(I - I') \cos i = I_1 \cos r;$$

[1] J. J. Thomson, *Phil. Mag.* Ap. 1880; Lorentz, *Schlömilch Zeitschrift*, vol. XXII.; Fitzgerald, *Phil. Trans.* 1880; Lord Rayleigh, *Phil. Mag.* Aug. 1881.

accordingly
$$\left.\begin{aligned} I' &= -\frac{I\sin(i-r)}{\sin(i+r)} \\ I_1 &= \frac{I\sin 2i}{\sin(i+r)} \end{aligned}\right\} \text{..........................(32),}$$

which are the same as Fresnel's formulæ. Since the light is refracted according to the ordinary law, these formulæ are true in the case of two isotropic media.

417. In the next place, let the light be polarized perpendicularly to the plane of incidence, so that the refracted ray is an extraordinary ray.

The conditions that the electric displacement perpendicular to the reflecting surface should be continuous, gives

$$(A + A')\sin i = A_1 \sin r \text{ .... (33).}$$

The condition that the electric forces parallel to this surface should be continuous, gives

$$V^2(A - A')\cos i = A_1 c^2 \cos r \text{ .... (34).}$$

Now if V_1 be the velocity of the extraordinary wave

$$\frac{I}{AV} = \frac{I'}{A'V} = \frac{I_1}{A_1 V_1},$$

also
$$V/V_1 = \sin i/\sin r,$$

and
$$V_1^2 = c^2\cos^2 r + a^2 \sin^2 r = p^2,$$

where p is the perpendicular from the point of incidence, on to the tangent plane to the ellipsoid, at the extremity of the extraordinary ray.

Equations (33) and (34) accordingly become

$$I + I' = I_1,$$

$$I - I' = \frac{I_1 c^2 \sin 2r}{p^2 \sin 2i},$$

whence
$$\left.\begin{aligned} I' &= \frac{I(p^2\sin 2i - c^2\sin 2r)}{p^2\sin 2i + c^2 \sin 2r} \\ I_1 &= \frac{2Ip^2\sin 2i}{p^2\sin 2i + c^2\sin 2r} \end{aligned}\right\} \text{............(35).}$$

The formulæ are the same as those furnished by MacCullagh's theory ; see § 253.

When the second medium is isotropic, $a=c=p$, whence (35) become

$$\left.\begin{aligned} I' &= \frac{I\tan(i-r)}{\tan(i+r)} \\ I_1 &= \frac{I\sin 2i}{\sin(i+r)\cos(i-r)} \end{aligned}\right\} \qquad (36),$$

which are the same as Fresnel's formulæ for light polarized perpendicularly to the plane of incidence.

418. Returning to (35), we observe that the intensity of the reflected light vanishes, when

$$p^2 \sin 2i = c^2 \sin 2r,$$

and since $$p\sin i = V\sin r,$$

this becomes $$V^2 \cot i = c^2 \cot r,$$

whence eliminating r, we obtain

$$\tan i = \frac{V(V^2-c^2)^{\frac{1}{2}}}{c(V^2-a^2)^{\frac{1}{2}}} \qquad (37),$$

which determines the polarizing angle.

419. Let us now suppose, that light polarized perpendicularly to the plane of incidence, is internally reflected at the surface of the crystal in contact with air. Then

$$a^2\sin^2 i + c^2\cos^2 i = V^2\sin^2 i \operatorname{cosec}^2 r,$$

whence $$\cos^2 r = \frac{c^2\cos^2 i - (V^2-a^2)\sin^2 i}{a^2\sin^2 i + c^2\cos^2 i},$$

and therefore since $V > a$, $\cos r$ will become imaginary, when

$$\tan i > \frac{c}{(V^2-a^2)^{\frac{1}{2}}}.$$

The right-hand side of this inequality determines the tangent of the critical angle, for light polarized perpendicularly to the plane of incidence.

Under these circumstances, the right-hand sides of (35) become complex, and it can be shown by the same methods as have already been employed, that the reflection is total, and is accompanied by a change of phase e, whose value is determined by the equation

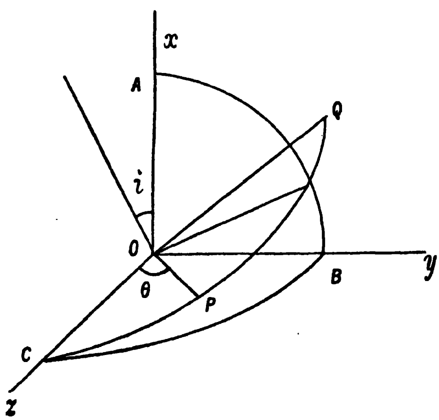
$$\tan \pi e/\lambda = V\{(V^2-a^2)\tan^2 i - c^2\}^{\frac{1}{2}}/c^2 \qquad (38).$$

When the second medium is isotropic, so that $a=c$, we fall back on Fresnel's formulæ.

The corresponding results for a uniaxal crystal cut parallel to the axis, may be obtained by interchanging a and c.

Crystalline Reflection and Refraction[1].

420. We shall now pass on to the general case in which the first medium is isotropic, whilst the second medium is a biaxal crystal.



Let O be the point of incidence, and let the axis of x be the normal to the reflecting surface, and let the axis of z coincide with the trace of the waves. Let COP be the front of the incident wave, OP, OQ the directions of the electric and magnetic displacements; also let $COP = \theta$, so that $COQ = \frac{1}{2}\pi + \theta$. Let COP', COP_1, COP_2 (not drawn in the figure) be the fronts of the reflected and the two refracted waves; i, r_1, r_2 the angles which the normals to the incident and two refracted waves make with the normal to the reflecting surface; also let OP', OP_1, OP_2 be the directions of the electric, and OQ', OQ_1, OQ_2 be those of the magnetic displacements in these waves.

Since the terms involving the suffix 2 are of the same form as those involving the suffix 1, they may be omitted during the work, and can be supplied at the end of the investigation.

[1] Glazebrook, *Proc. Camb. Phil. Soc.* vol. IV. p. 155.

The continuity of electric displacement along OA, gives

$$A \cos AP + A' \cos AP' = A_1 \cos AP_1 \quad \text{..........(39)}.$$

The continuity of electric force parallel to OB and OC, give

$$V^2 A \cos BP + V^2 A' \cos BP' = V_1^2 A_1 \cos BP_1 + V^2 A_1 \tan \chi_1 \sin r_1 \quad \text{............(40)},$$

and

$$V^2 A \cos CP + V^2 A' \cos CP' = V_1^2 A_1 \cos CP_1 \quad \text{..................(41)},$$

where χ_1 is the angle between the refracted ray and the wave normal.

The continuity of magnetic induction along OA, gives

$$VA \cos AQ + VA' \cos AQ' = V_1 A_1 \cos AQ_1 \quad \text{......(42)}.$$

The continuity of magnetic force, parallel to OB and OC, give

$$VA \cos BQ + VA' \cos BQ' = V_1 A_1 \cos BQ_1 \quad \text{......(43)},$$

and $$VA \cos CQ + VA' \cos CQ' = V_1 A_1 \cos CQ_1 \quad \text{......(44)}.$$

Now $$\frac{V}{\sin i} = \frac{V_1}{\sin r_1},$$

also $$\cos AP = \sin i \sin \theta, \quad \cos CQ = -\sin \theta;$$

whence (39) and (44) reduce to

$$(A \sin \theta + A' \sin \theta') \sin i = A_1 \sin r_1 \sin \theta_1 \quad \text{.........(45)},$$

which proves the equivalence of (39) and (44).

Again $$\cos CP = \cos \theta, \quad \cos AQ = \sin i \cos \theta;$$

whence (41) and (42) reduce to

$$(A \cos \theta + A' \cos \theta') \sin^2 i = A_1 \cos \theta_1 \sin^2 r_1 \quad \text{........(46)}.$$

Since $$\cos BP = \cos i \sin \theta, \quad \cos BQ = \cos i \cos \theta,$$

(40) and (43) become,

$$(A \sin \theta - A' \sin \theta') \sin^2 i \cos i = A_1 (\cos r_1 \sin \theta_1 + \sin^2 r_1 \tan \chi_1) \sin^2 r \quad \text{...........(47)},$$

and

$$(A \cos \theta - A' \cos \theta') \sin i \cos i = A_1 \sin r_1 \cos r_1 \quad \text{..................(48)}.$$

Recollecting that if I, I', I_1 are the square roots of the intensities,

$$\frac{I}{A \sin i} = \frac{I'}{A' \sin i} = \frac{I_1}{A_1 \sin r_1},$$

and restoring the terms in A_2, we finally obtain from (46), (48), (45) and (47)

$$\left.\begin{aligned}(I\cos\theta + I'\cos\theta')\sin i &= I_1\cos\theta_1\sin r_1 + I_2\cos\theta_2\sin r_2\\ (I\cos\theta - I'\cos\theta')\cos i &= I_1\cos\theta_1\cos r_1 + I_2\cos\theta_2\cos r_2\\ I\sin\theta + I'\sin\theta' &= I_1\sin\theta_1 + I_2\sin\theta_2\\ (I\sin\theta - I\sin\theta')\sin 2i &= I_1(\sin\theta_1\sin 2r_1 + 2\sin^2 r_1\tan\chi_1)\\ &\quad + I_2(\sin\theta_2\sin 2r_2 + 2\sin^2 r_2\tan\chi_2)\end{aligned}\right\} \quad (49).$$

When the angle of incidence is given, θ_1, θ_2, r_1, r_2, χ_1, χ_2 are known from the properties of the wave surface, hence these equations are sufficient to determine the unknown quantities I', I_1, I_2 and θ'.

421. Equations (49) are the same as those obtained by means of Lord Kelvin's modification of Lord Rayleigh's theory—see (33) of § 270; and it is also remarkable, that they are the same as those obtained in 1835 by MacCullagh[1] by means of an erroneous theory. MacCullagh discussed these equations, and compared the results obtained from them with the experiments of Brewster[2], and found that they agreed fairly well. Accordingly, although we cannot at the present time accept the assumptions, upon which MacCullagh based his theory, as sound, yet most of the results of his first paper, with certain modifications necessitated by his having supposed that the vibrations of polarized light are parallel to the plane of polarization, are applicable to the electromagnetic theory; and thus MacCullagh's investigations regain their interest.

422. The discussion of (49) may be facilitated by a device invented by MacCullagh[3].

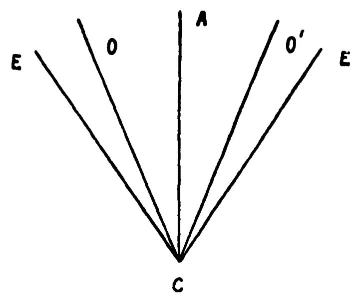
When polarized light is incident upon a crystalline reflecting surface at a given angle, it is known both from experiment and theory, that it is always possible by properly choosing the plane of polarization of the incident light, to make one or other of the two refracted rays disappear. The two directions of vibration for which this is possible, are called by MacCullagh *uniradial di-*

[1] *Trans. Roy. Irish Acad.* vols. XVIII. p. 31 and XXI. p. 17.

[2] *Phil. Trans.* 1819, p. 145; Seebeck, *Pogg. Ann.* vol. XXI. p. 290; XXII. p. 126; XXXVIII. p. 230; Glazebrook, *Phil. Trans.* 1879, p. 287; 1880, p. 421.

[3] *Trans. Roy. Irish Acad.* vol. XVIII. p. 31.

rections. In the figure, let CA be the line of intersection of the plane of incidence with the plane of the paper, and let CO be the



direction of vibration of the incident light, when the ordinary ray alone exists, and CE the corresponding direction when the extraordinary ray alone exists; also let CO', CE' be the directions of the vibrations in the reflected waves corresponding to CO and CE.

Now whatever may be the character of the incident light, the vibrations may always be conceived to be resolved along the two uniradial directions CO, CE; and these two vibrations will give rise to the vibrations CO', CE' in the reflected wave. If the incident light is plane polarized, the vibrations CO, CE, and also the vibrations CO', CE' will be in the same phase, and therefore the reflected light will be plane polarized, although its plane of polarization will not usually coincide with that of the incident light. If however the incident light be not plane polarized, the phases of the vibrations CO, CE, and therefore of CO', CE' will be different; hence the reflected light will not usually be plane polarized. It is however usually possible by properly choosing the angle of incidence, to make the two reflected vibrations CO', CE' coincide; and whenever this is possible, the reflected light will be plane polarized, and the angle of incidence at which this takes place is therefore the polarizing angle. These considerations, as we shall presently show, greatly simplify the problem of finding the polarizing angle.

423. The four vibrations CO, CE, CO', CE' do not usually lie in the same plane; we can however show, *that when one of the refracted rays is absent, the lines of intersection of the planes of polarization of the three waves with their respective wave-fronts lie in a plane.*

Let CP, CP', CP_1 be the lines of intersection of the three planes of polarization with their respective wave-fronts; λ, μ, ν; λ', μ', ν'; λ_1, μ_1, ν_1 the direction cosines of CP, CP', CP_1.

Then

$$\begin{array}{lll} \lambda = \cos\theta \sin i, & \mu = \cos\theta\cos i, & \nu = -\sin\theta, \\ \lambda' = \cos\theta' \sin i, & \mu' = -\cos\theta' \cos i, & \nu' = -\sin\theta', \\ \lambda_1 = \cos\theta_1 \sin r_1, & \mu_1 = \cos\theta_1 \cos r_1, & \nu_1 = -\sin\theta_1. \end{array}$$

Putting $I_2 = 0$ in the first three of (49) and substituting, we obtain

$$\begin{aligned} I\lambda + I'\lambda' &= I_1\lambda_1, \\ I\mu + I'\mu' &= I_1\mu_1, \\ I\nu + I'\nu' &= I_1\nu_1, \end{aligned}$$

whence

$$\begin{vmatrix} \lambda, & \lambda', & \lambda_1 \\ \mu, & \mu', & \mu_1 \\ \nu, & \nu', & \nu_1 \end{vmatrix} = 0,$$

which is the condition that CP, CP', CP_1 should lie in the same plane.

The preceding theorem is a modification of one due to MacCullagh.

424. Let us now suppose, that the reflecting surface is a uniaxal crystal, and let the suffixes 1 and 2 refer to the ordinary and extraordinary rays respectively; then $\chi_1 = 0$. If we suppose that the ordinary ray alone exists, $I_2 = 0$, and we easily obtain from (49) the equations

$$\left.\begin{aligned} \tan\theta &= \cos(i - r_1)\tan\theta_1 \\ \tan\theta' &= -\cos(i + r_1)\tan\theta_1 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots \quad (50).$$

Since the angle of incidence is supposed to be given, r_1 and θ_1 are known; and therefore (50) determine θ, θ' which give the directions of vibration in the incident and reflected waves.

Again, suppose that the extraordinary ray alone exists; putting $I_1 = 0$, and writing Θ, Θ' for θ, θ', we obtain

$$\left.\begin{aligned} \tan\Theta &= \cos(i - r_2)\tan\theta_2 + \frac{\sin^2 r_2 \tan\chi}{\cos\theta_2 \sin(i + r_2)} \\ \tan\Theta' &= -\cos(i + r_2)\tan\theta_2 + \frac{\sin^2 r_2 \tan\chi}{\cos\theta_2 \sin(i - r_2)} \end{aligned}\right\} \quad \ldots \quad (51),$$

which determine Θ and Θ'.

Now we have shown that in order that the reflected light should be plane polarized, it is necessary that the two directions CO', CE' should coincide, in which case $\theta' = \Theta'$; we thus obtain from (50) and (51),

$$\cos(i + r_1)\tan\theta_1 - \cos(i + r_2)\tan\theta_2 + \frac{\sin^2 r_2 \tan\chi}{\cos\theta_2 \sin(i - r_2)} = 0 \quad (52),$$

which determines the polarizing angle i.

425. A very elegant formula is given by MacCullagh for the polarizing angle, when the plane of incidence contains the axis of a uniaxal crystal, which is most simply obtained directly from (49), by determining the angle of incidence at which the intensity of the reflected light vanishes, when the incident light is polarized perpendicularly to the plane of incidence.

We have $\quad I' = I_1 = 0, \quad \theta = \theta' = \theta_2 = \frac{1}{2}\pi$;

also if ω is the angle which the extraordinary wave normal makes with the axis of the crystal

$$\tan\chi_2 = \frac{(a^2 - c^2)\sin\omega\cos\omega}{a^2\sin^2\omega + c^2\cos^2\omega} = \frac{(a^2 - c^2)\sin\omega\cos\omega\sin^2 i}{V^2\sin^2 r} \quad \text{...(53)},$$

where c and V are the velocities of propagation of the ordinary wave within the crystal, and in the medium surrounding the crystal, and $r = r_2$.

From the third of (49) we obtain $I = I_2$, and from the last

$$\begin{aligned} \sin 2i - \sin 2r &= 2\sin^2 r \tan\chi_2 \\ &= V^{-2}(a^2 - c^2)\sin 2\omega\sin^2 i \quad \text{.........(54)}, \end{aligned}$$

by (53).

If λ be the angle which the optic axis makes with the reflecting surface, $\omega + \lambda = \frac{1}{2}\pi - r$; whence multiplying (54) by $\tan r$, we obtain

$$\sin^2 r = \sin i\cos i\tan r - V^{-2}(a^2 - c^2)\sin(r + \lambda)\cos(r + \lambda)\sin^2 i\tan r.$$

But $\quad \sin^2 r = V^{-2}\sin^2 i\,\{c^2 + (a^2 - c^2)\cos^2(r + \lambda)\}$.

Equating these two values of $\sin^2 r$, and reducing we shall obtain

$$\tan r = \frac{a^2\cos^2\lambda + c^2\sin^2\lambda}{V^2\cot i + (a^2 - c^2)\sin\lambda\cos\lambda}.$$

Substituting in (54), and reducing, we finally obtain

$$\sin^2 i = \frac{V^2(V^2 - a^2\cos^2\lambda - c^2\sin^2\lambda)}{V^4 - a^2c^2} \quad \text{.........} \quad (55),$$

which is the formula in question, which determines the polarizing angle i.

Reflection at a Twin Plane.

426. We shall conclude this Chapter by giving an account of a peculiar kind of reflection, which is produced by iridescent crystals of chlorate of potash.

The phenomena exhibited by the crystals in question were first examined by Sir G. Stokes[1], and the experimental results at which he arrived may be summed up as follows:—

(i) If one of the crystalline plates be turned round in its own plane, without altering the angle of incidence, the peculiar reflection vanishes twice in a revolution, viz. when the plane of incidence coincides with the plane of symmetry of the crystal.

(ii) As the angle of incidence increases, the reflected light becomes brighter, and rises in refrangibility.

(iii) The colours are not due to absorption, the refracted light being strictly complementary to the reflected.

(iv) The coloured light is not polarized. It is produced indifferently, whether the incident light be common light, or light polarized in any plane; and is seen, whether the reflected light be viewed directly, or through a Nicol's prism turned in any way.

(v) The spectrum of the reflected light is frequently found to consist almost entirely of a comparatively narrow band. When the angle of incidence is increased, the band moves in the direction of increasing refrangibility, and at the same time increases rapidly in width. In many cases the reflection appears to be almost total.

427. Sir G. Stokes has shown that the seat of the colour is a narrow layer about a thousandth of an inch in thickness, and he suggested that this layer consists of a twin stratum. The subject was subsequently taken up by Lord Rayleigh, who attributed the phenomena to the existence of a number of twin planes in contact with one another; and he has accounted for most of the phenomena by means of the electromagnetic theory of light. He has also shown, both from theory and experiment, that when the angle of incidence is sufficiently small, and the planes of incidence

[1] On a remarkable Phenomenon of Crystalline Reflection, *Proc. Roy. Soc.*, Feb. 26, 1885; see also Lord Rayleigh, *Phil. Mag.* Sep. 1888, p. 256; *Proc. Roy. Institution*, 1889.

and symmetry are perpendicular, reflection at a twin plane reverses the polarization; that is to say, if the incident light is polarized in the plane of incidence, the reflected light is polarized in the perpendicular plane and *vice versâ*. This very peculiar law was not even suspected, until it had been obtained by theoretical considerations.

428. The easiest way of understanding what is meant by a twin-crystal, is to suppose that a crystal of Iceland spar is divided into two portions by a plane, which is inclined at any angle α to the optic axis, and that one portion is turned through two right angles. The optic axes of the two portions will still lie in the same plane, but instead of being coincident, they will be inclined to one another at an angle 2α. Crystals whose structure is of this character, are called twin-crystals; and it is evident, that a crystal may possess more than one twin layer.

429. We shall now consider Lord Rayleigh's theory[1].

When the plane of incidence contains the optic axes of the two portions, and the light is polarized in the plane of incidence, the wave surfaces in both crystals are spheres of equal radii; and therefore the crystal will act like two isotropic media, whose optical properties are identical. Hence no reflection can take place, and the wave will pass on undisturbed.

430. We shall in the next place suppose, that the light is polarized perpendicularly to the plane of incidence.

Let the axis of x be normal to the twin plane, and let the plane xy contain the optic axes of the two portions. Let Oy' be the axis of the upper portion, and let Ox' be perpendicular to Oy' in the plane xy. Let $x'Ox = \alpha$; let f', g' be the electric displacements along Ox', Oy'; and let f, g, h be the displacements along Ox, Oy, Oz.

In the upper portion, the wave surface for the extraordinary ray consists of the planetary ellipsoid

$$x'^2/c^2 + (y'^2 + z^2)/a^2 = 1.$$

Accordingly by (5) we have

$$\left.\begin{aligned} P &= 4\pi c^2 f' \cos\alpha + 4\pi a^2 g' \sin\alpha \\ Q &= -4\pi c^2 f' \sin\alpha + 4\pi a^2 g' \cos\alpha \\ R &= 4\pi a^2 h \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots (56).$$

[1] *Phil. Mag.* Sep. 1888, p. 241.

But $\quad f' = f\cos\alpha - g\sin\alpha,\quad g' = f\sin\alpha + g\cos\alpha,$

whence if
$$A = (a^2\sin^2\alpha + c^2\cos^2\alpha),$$
$$C = (a^2\cos^2\alpha + c^2\sin^2\alpha),$$
$$B = (a^2 - c^2)\sin\alpha\cos\alpha,$$
$$D = a^2,$$

(56) become
$$P = 4\pi(Af + Bg),\quad Q = 4\pi(Bf + Cg),\quad R = 4\pi Dh \;\ldots\; (57).$$

The equations of electric force for the lower medium are obtained by changing the sign of α, whence
$$P_1 = 4\pi(Af_1 - Bg_1),\quad Q_1 = 4\pi(-Bf_1 + Cg_1),\quad R_1 = 4\pi Dh_1 \ldots (58).$$

Since none of the quantities are functions of z, it follows that if we substitute these values in (11), and put $\mu = 1$, and recollect that $h = h_1 = 0$, we obtain
$$\left.\begin{aligned}\frac{d^2f}{dt^2} &= \frac{d^2}{dy^2}(Af + Bg) - \frac{d^2}{dx\,dy}(Bf + Cg)\\ \frac{d^2g}{dt^2} &= \frac{d^2}{dx^2}(Bf + Cg) - \frac{d^2}{dx\,dy}(Af + Bg)\end{aligned}\right\}\ldots\ldots\ldots (59).$$

Let the displacements in the incident, reflected, and refracted waves be
$$\left.\begin{aligned} f &= qS, & g &= -pS\\ f' &= qA'S', & g' &= -p'A'S'\\ f_1 &= qA_1S_1, & g_1 &= -p_1A_1S_1\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots (60)$$

where
$$S = \epsilon^{\iota(px + qy - st)},$$

and S' and S_1 are obtained by changing p into p' and p_1 respectively. Since p and q are proportional to the direction cosines of the incident wave, these equations satisfy the conditions that the displacement lies in the front of the wave.

Substituting from the first of (60) in (59), we find that both equations lead to
$$s^2 = Aq^2 - 2Bpq + Cp^2 \;\ldots\ldots\ldots\ldots\ldots\ldots (61),$$

which is a quadratic equation for determining the two values of p corresponding to a given value of s.

Changing the sign of B, we find for the second medium
$$s^2 = Aq^2 + 2Bp_1q + Cp_1^2 \ldots\ldots\ldots\ldots\ldots\ldots (62).$$

Equating the two values of s, we obtain
$$C(p^2 - p_1^2) = 2Bq(p + p_1),$$

or
$$C(p - p_1) = 2Bq \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots (63).$$

We have now to express the boundary conditions.

The condition of continuity of electric displacement perpendicular to the twin surface gives

$$1 + A' = A_1.$$

The condition that the electric forces are continuous gives

$$Bq - Cp + A'(Bq - Cp') = -A_1(Bq + Cp_1).$$

Eliminating A_1, we obtain

$$A'(2Bq - Cp' - Cp_1) = C(p - p_1) - 2Bq = 0$$

by (63); whence $A' = 0$.

431. This result shows, that when the light is polarized perpendicularly to the plane of incidence, the amplitude of the reflected light is zero. Accordingly when light of any kind is incident upon the crystal, the light reflected at the twin-plane vanishes, when the plane of incidence is a plane of symmetry. Accordingly under these circumstances, the reflected light is entirely produced by the outer surface of the crystal, and is unaffected by the existence of the twin plane.

432. We shall next consider the case, in which the plane of incidence is perpendicular to the plane of symmetry.

In this case, none of the quantities are functions of y, and we may accordingly write for the incident wave

$$f = S\lambda, \quad g = S\mu, \quad h = S\nu,$$

where

$$S = \epsilon^{\iota(px + rz - st)};$$

but since $\lambda p + \nu r = 0$, these may be written

$$f, g, h = (r, \mu, -p)S.$$

By (5) and (57), the equations of motion for the upper medium are

$$\frac{d^2f}{dt^2} = \frac{d^2}{dz^2}(Af + Bg) - D\frac{d^2h}{dx\,dz},$$

$$\frac{d^2g}{dt^2} = \left(\frac{d^2}{dx^2} + \frac{d^2}{dz^2}\right)(Bf + Cg),$$

$$\frac{d^2h}{dt^2} = D\frac{d^2h}{dx^2} - \frac{d^2}{dx\,dz}(Af + Bg).$$

Substituting the above values of f, g, h, the first and third give

$$s^2 = r(Ar + B\mu) + p^2D \quad\ldots\ldots\ldots\ldots\ldots\ldots (64),$$

whilst the second gives

$$\mu s^2 = (p^2 + r^2)(Br + C\mu) \quad\ldots\ldots\ldots\ldots\ldots\ldots (65).$$

433. These equations determine p and μ, when r and s, which are the same for all the waves, are given; accordingly if we eliminate μ from (64) and (65) we shall obtain a quadratic in p^2, the roots of which may be written $\pm p_1$, $\pm p_2$, where p_1, p_2 are positive quantities. If therefore a wave polarized in any azimuth be incident from an isotropic medium upon the upper face of the first twin, the two refracted waves will be determined by the values of p_1, p_2, and their direction cosines by the equations $l_1/p_1 = n_1/r$, $l_2/p_2 = n_2/r$. These two waves, when incident on the twin plane, will give rise to two reflected waves in the first twin, and two refracted waves in the second; and the directions of the two former are given by $l_1/p_1 = -n_1/r$ and $l_2/p_2 = -n_2/r$. If the azimuth of the plane of polarization is such, that there is only one refracted wave, say p_1, in the first twin, there would still be two reflected waves, whose directions are determined by the preceding equations; and it is worthy of notice that the angle of reflection of one of these waves is not equal to the angle of incidence, but is equal to the angle of incidence of the other wave p_2. We shall also denote the values of μ corresponding to p_1, p_2 by μ_1, μ_2.

With regard to the two refracted waves in the second twin, we see from (57) and (58), that the sign of B must be changed. This will make no difference in the values of p_1, p_2, but will change the signs of μ_1, μ_2; so that in the second twin, the values of μ corresponding to p_1, p_2 are $-\mu_1$, $-\mu_2$.

434. We are now in a position to find the intensities of the light reflected at the surface of separation.

Let the incident wave be

$$f, g, h = (r, \mu_1, -p_1)\,\Theta_1 \epsilon^{\iota(p_1x+rz-st)} + (r, \mu_2, -p_2)\,\Theta_2 \epsilon^{\iota(p_2x+rz-st)},$$

then the reflected waves will be

$$f', g', h' = (r, \mu_1, p_1)\,A' \epsilon^{\iota(-p_1x+rz-st)} + (r, \mu_2, p_2)\,A'' \epsilon^{\iota(-p_2x+rz-st)},$$

and the refracted waves

$$f_1, g_1, h_1 = (r, -\mu_1, -p_1)\,A_1 \epsilon^{\iota(p_1x+rz-st)} + (r, -\mu_2, -p_2)\,A_2 \epsilon^{\iota(p_2x+rz-st)}.$$

The continuity of f at the twin plane requires that

$$\Theta_1 + \Theta_2 + A' + A'' = A_1 + A_2 \ldots\ldots\ldots\ldots\ldots (66).$$

The continuity of electric force parallel to z and y give

$$p_1\Theta_1 + p_2\Theta_2 - p_1A' - p_2A'' = p_1A_1 + p_2A_2 \ldots\ldots\ldots (67),$$

and

$$(Br + C\mu_1)(\Theta_1 + A' + A_1) + (Br + C\mu_2)(A'' + A_2 + \Theta_2) = 0 \ldots (68).$$

Since $dc/dt = dQ/dx$, the continuity of magnetic force parallel to z requires, that dQ/dx should be continuous; whence

$$p_1(Br + C\mu_1)(\Theta_1 - A' + A_1) + p_2(Br + C\mu_2)(\Theta_2 - A'' + A_2) = 0 \quad \ldots (69).$$

If V be the velocity of any wave,

$$V^2 = s^2/(p^2 + r^2),$$

and therefore by (65)

$$V^2\mu = Br + C\mu.$$

Writing $\qquad p_2/p_1 = \varpi, \quad \mu_2 V_2^2/\mu_1 V_1^2 = \sigma$

equations (66) to (69) become

$$\Theta' + \Theta_2 + A' + A'' = \quad A_1 + A_2,$$

$$\Theta_1 + \varpi\Theta_2 - A' - \varpi A'' = \quad A_1 + \varpi A_2,$$

$$\Theta_1 + \sigma\Theta_2 + A' + \sigma A'' = -A_1 - \sigma A_2,$$

$$\Theta_1 + \sigma\varpi\Theta_2 - A' - \sigma\varpi A'' = -A_1 - \sigma\varpi A_2.$$

Solving these equations we obtain

$$A' = -\frac{(\varpi - 1)\sigma}{(\varpi - \sigma)(\varpi\sigma - 1)}\{(1 + \varpi)\Theta_1 + \varpi(1 + \sigma)\Theta_2\} \ldots (70),$$

$$A'' = \frac{\varpi - 1}{(\varpi - \sigma)(\varpi\sigma - 1)}\{(1 + \sigma)\Theta_1 + \sigma(1 + \varpi)\Theta_2\} \ldots (71).$$

These equations are perfectly general, but when the doubly-refracting power of the twin is small, they may be simplified. In this case p_1, p_2, V_1, V_2 are very nearly equal, and we may write $\varpi = 1$, $\sigma = \mu_2/\mu_1$, and (70) and (71) become

$$A' = \frac{p_2 - p_1}{p(\mu_2 - \mu_1)^2}\{2\mu_1\mu_2\Theta_1 + \mu_2(\mu_1 + \mu_2)\Theta_2\} \ldots\ldots (72),$$

$$A'' = \frac{p_1 - p_2}{p(\mu_2 - \mu_1)^2}\{\mu_1(\mu_1 + \mu_2)\Theta_1 + 2\mu_1\mu_2\Theta_2\} \ldots\ldots (73).$$

435. We shall now suppose, that the amplitudes of the two components of the light incident upon the first face of the crystal are M, perpendicular to, and N in the plane of incidence. Then since the doubly-refracting power is supposed to be small, it follows, that as the right-hand sides of (72) and (73) involve the factor $p_1 - p_2$, we may neglect the slight loss of light due to refraction, and take

$$\left.\begin{aligned} M &= \mu_1\Theta_1 + \mu_2\Theta_2 \\ N &= (p^2 + r^2)^{\frac{1}{2}}(\Theta_1 + \Theta_2) \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (74).$$

Similarly if M', N' be the amplitudes of the emergent vibrations, we may take

$$\left.\begin{aligned} M' &= \mu_1 A' + \mu_2 A'' \\ N' &= (p^2+r^2)^{\frac{1}{2}}(A' \mp A'') \end{aligned}\right\} \quad\text{..................(75).}$$

If the twin stratum is *thin*, we may, as a first approximation, substitute the values of A', A'' from (72) and (73) in (75), and we shall obtain

$$\begin{aligned} M' &= \frac{(p_2-p_1)\,\mu_1\mu_2}{p\,(\mu_1-\mu_2)}(\Theta_1+\Theta_2) \\ &= \frac{(p_2-p_1)\,\mu_1\mu_2 N}{p\,(\mu_1-\mu_2)(p^2+r^2)^{\frac{1}{2}}} \quad\text{.................. (76),} \end{aligned}$$

and
$$N' = -\frac{(p_2-p_1)(p^2+r^2)^{\frac{1}{2}} M}{p\,(\mu_1-\mu_2)} \quad\text{.............. (77).}$$

Eliminating s between (64) and (65), we obtain

$$Br\mu^2 + \{(A-C)\,r^2 + (D-C)\,p^2\}\,\mu - Br\,(p^2+r^2) = 0,$$

from which we obtain

$$-\mu_1\mu_2 = p^2 + r^2.$$

From (64) we also obtain

$$Br\,(\mu_1-\mu_2) + (p_1^2 - p_2^2)\,D = 0,$$

whence
$$\frac{p_1-p_2}{\mu_1-\mu_2} = -\frac{rB}{2pD}.$$

Accordingly (76) and (77) become

$$\left.\begin{aligned} M' &= -\frac{r\,(p^2+r^2)^{\frac{1}{2}} BN}{2p^2 D} \\ N' &= -\frac{r\,(p^2+r^2)^{\frac{1}{2}} BM}{2p^2 D} \end{aligned}\right\} \quad\text{..................(78).}$$

These equations show, that the intensity of the light which emerges from the upper surface, after having been reflected at the twin plane, is proportional to that of the incident light, without regard to the polarization of the latter. If the incident light is unpolarized, which occurs when M and N are equal, and without any permanent phase relation, so is also the emergent light; *also if the incident light is polarized in or perpendicularly to the plane of incidence,* the emergent light is polarized in the opposite manner.

436. The preceding results are true only as a first approximation, when the double refracting power is small, and the twin stratum is thin; and by proceeding to a higher degree of approxi-

mation, Lord Rayleigh has shown, that the reversal of the polarization will only take place, when the angle of incidence is small. This agrees with experiment.

Metallic Reflection.

437. We stated in Chapter XVIII., § 385, that metallic reflection could not be satisfactorily accounted for on the electromagnetic theory, by taking into account the conductivity.

When the conductivity is introduced, we obtain from (6), (7) and (5)

$$u = \dot{f} + CP = \dot{f} + 4\pi Cf/K;$$

accordingly for an isotropic medium, the general equations of electric displacement become

$$\frac{d^2f}{dt^2} + \frac{4\pi C}{K}\frac{df}{dt} = \frac{1}{\mu K}\nabla^2 f.$$

These equations are of a very similar form to the equations of motion of an elastic medium, into which a viscous term has been introduced, and by integrating them in the usual manner, it can be shown that the square of the pseudo-refractive index (that is $\sin^2 i/\sin^2 r$) must be a complex quantity, *whose real part must be positive.*

NOTE TO § 402.

Mr Larmor has pointed out the following additional results:

Velocity of Light.			*Ratio of Electric Units.*		
Cornu	(1878)	29985×10^6	Himstedt	(1888)	30076×10^6
Michelson	(1882)	29986×10^6	Klemençic	(1887)	30150×10^6
Newcomb	(1882)	30040×10^6	Lord Kelvin	(1889)	30040×10^6

See also J. J. Thomson, *Proc. Roy. Soc.* 1890.

CHAPTER XX.

ACTION OF MAGNETISM ON LIGHT.

438. THE electromagnetic theory of light. so far as it has been developed in the preceding chapter, depends upon the hypothesis that a medium exists, whose special function is to propagate electrostatic and electromagnetic effects; and that when electromagnetic waves, whose periods lie between certain limits, are transmitted through the medium, the sensation of light is produced. If therefore light is the effect of an electromagnetic disturbance, the natural inference is, that an intimate connection exists between electricity and light; and that when a wave of light passes through an electromagnetic field, it ought to experience certain modifications during its passage, and to emerge from the field in a different condition from that in which it entered.

439. The conviction that a direct relation exists between electricity and light, led Faraday to attempt many experiments for the purpose of discovering some mutual action between the two classes of phenomena; but it was not until 1845, that he made the important discovery, that a field of magnetic force possesses the power of *rotating the plane of polarization of light*. During recent years, much attention has been devoted to this subject, and numerous experiments have been made by Kerr, Kundt and others, which have greatly extended our knowledge. We shall therefore commence by giving an account of the principal experimental results, and then proceed to enquire, how far they may be explained by theoretical considerations.

Faraday's Experiments.

440. The first experiment described by Faraday[1], consisted in placing a plate composed of a variety of heavy glass, called silicated borate of lead, between the poles of an electromagnetic; and he found that when a ray of plane polarized light was transmitted through the glass in the direction of the lines of magnetic force, the plane of polarization was rotated in the *same* direction as that of the amperean current which would produce the force.

441. Further experiments upon a variety of other transparent media led to the following law:—*In diamagnetic substances the direction of rotation of the plane of polarization is positive; that is to say, it is in the same direction as a positive current must circulate round the ray, in order to produce a magnetic force in the same direction, as that which actually exists in the medium.*

The amount of rotation depends upon the nature of the medium and the strength of the magnetic force. No rotation has been observed, when the magnetic force is perpendicular to the direction of the ray.

442. Verdet[2] however discovered, that certain ferromagnetic media, such as a strong solution of perchloride of iron in wood spirit or ether, produced a rotation in the opposite direction to that of the current, which would give rise to the magnetic force.

Kerr's Experiments.

443. Between the years 1875 and 1880, two very important series of experiments were made by Dr Kerr of Glasgow, upon the connection between light and electricity. The first series relate to the effect of electrostatic force, and the second to magnetic force.

Experiments on the Effect of Electrostatic Force.

444. In these experiments[3], a transparent dielectric was subjected to the action of electrostatic force, and the effect of the latter upon light was observed.

[1] *Phil. Trans.* 1845, p. 1; *Exp. Res.* XIXth series §§ 2146—2242.
[2] *Ante*, p. 159.
[3] *Phil. Mag.* Nov. 1875, p. 339.

We shall first consider the case in which the dielectric is a plate of glass placed in a vertical plane, and shall suppose that the electrostatic force is horizontal.

Polarized light was transmitted at normal incidence through the plate of glass, and the analyser was placed in the position of extinction; and it was found, that when an electrostatic force was made to act upon the dielectric, the light reappeared, and disappeared after the force was removed. The effect was most marked, when the plane of polarization of the incident light was inclined at an angle of about 45° to the force; but when the incident light was polarized in or perpendicularly to the direction of the force, no effect was observed.

The light restored by electrostatic action was elliptically polarized, and could not therefore be extinguished by any rotation of the analyser.

It was also found that the optical effect was independent of the direction of the force; that is to say, its intensity remained unchanged when the direction of the force was reversed.

The optical effect did not acquire its maximum intensity at the instant the force commenced to act, but gradually increased during a period of about thirty seconds, at the end of which it attained its full effect. Also when the force was removed, the effect did not immediately disappear, but faded away at first rapidly, and then more gradually to perfect extinction.

445. It is known that compressed glass acts like a negative uniaxal crystal, whose axis is parallel to the direction of compression; whilst stretched glass acts like a positive uniaxal crystal, whose axis is parallel to the direction of extension. Accordingly Kerr introduced a slip of glass, called a compensator, and found that when the slip was compressed in a direction parallel to the lines of electrostatic force, the optical effect produced by the latter was strengthened, but when the slip was stretched in that direction, the effect was weakened.

From these experiments Kerr concluded, that the effect of electrostatic stress on glass is to transform it into a medium, which possesses the optical properties of a negative uniaxal crystal, whose axis is parallel to the direction of the force. Under these circumstances, it ought to follow, that glass, when under the action of electrostatic force, should be capable of producing the

rings and brushes of uniaxal crystals, but no experiments elucidating this point appear to have been performed.

446. From experiments made on resin, it appeared that the effect of electrostatic force upon this substance was to convert it into a medium, which is optically equivalent to a positive uniaxal crystal.

447. A few experiments were made on a plate of quartz, whose axis was perpendicular to the direction of the force; and these experiments indicated, that the optical effects were of a similar kind to those produced upon glass. It is to be hoped, that more elaborate experiments upon quartz will be attempted; for if the optical effects are of a similar kind to those produced upon glass, it would follow that the effect of electrostatic force would be to convert a plate of quartz, whose axis is perpendicular to the force, into a biaxal crystal, which is capable of producing rotatory polarization. Since the principal wave velocity in the direction of the force, is capable of being varied at pleasure by increasing or diminishing the force, we should anticipate that some very curious phenomena would be observed in connection with coloured rings, and also possibly in connection with conical refraction.

448. Experiments made on liquids[1] showed, that disulphide of carbon, benzol, paraffin and kerosin oils, and spirits of turpentine act, when subjected to electrostatic force, like a positive uniaxal crystal, whose axis is parallel to the direction of the force; whilst olive oil acts like a negative uniaxal crystal. Turpentine, as is known, produces rotatory polarization, some specimens being right-handed and others left-handed; and therefore in experimenting upon this liquid, two samples of contrary photogyric power were mixed together, in such proportions as to destroy the rotatory properties of the mixture.

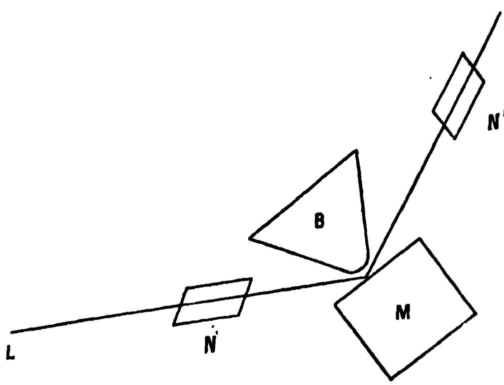
449. Further experiments[2] led to the following law:—*The effect of electrostatic force upon an isotropic transparent dielectric, is to render it optically equivalent to a uniaxal crystal, whose axis is parallel to the direction of the force; and the difference between the retardations of the ordinary and extraordinary rays, is proportional to the product of the thickness of the dielectric, and the square of the resultant electric force.*

[1] *Phil. Mag.* December, 1875, p. 446.
[2] *Ibid.* March, 1880, p. 157.

Kerr's Experiments on Reflection from a Magnet.

450. Shortly after the experiments described in the preceding sections had been made, Kerr commenced a series of experiments upon light reflected from an electromagnet. In the first series of experiments, the light was reflected from the pole of the magnet; whilst in the second series, a bar of soft iron was laid across the poles of an electromagnet, so that the lines of magnetic force were parallel to the reflecting surface.

451. We shall now describe the first series of experiments and the apparatus employed[1].



An electromagnet *M*, consisting of a solid core of soft iron surrounded by a wire making 400 turns, was worked by a Grove's battery of six cells; and the poles of the magnet were carefully polished, so as to form a good reflecting surface. The source of light was a narrow paraffin flame *L*, which was polarized by a Nicol *N*, and the reflected light was analysed by a second Nicol *N'*. A wedged-shape piece of soft iron *B* with a well-rounded edge, called a submagnet, was placed in close proximity to the reflecting surface, with its rounded edge perpendicular to the plane of incidence, so as to leave a space of about $\frac{1}{20}$th of an inch between the two. The object of the submagnet was to intensify the magnetic force in the neighbourhood of the mirror, when the circuit was closed; and Kerr found that without it, he never obtained any optical effect. The preceding arrangement was employed, when the angle of incidence lay between 60° and 80°; but when the incidence was perpendicular, a different arrangement was adopted, which will be explained later on.

[1] *Phil. Mag.* May, 1887.

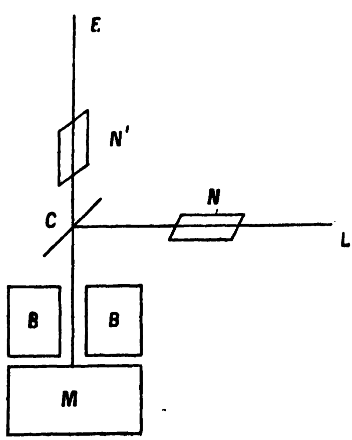
452. *Experiment I.* Light polarized in or perpendicularly to the plane of incidence is allowed to fall on the pole of the electro-magnet, and the analyser is placed in the position of extinction. When the circuit is closed, so that the reflector becomes magnetized, the light immediately reappears; when the circuit is broken, the light disappears, and again reappears when the current is reversed.

The light reflected whilst the circuit is closed is elliptically polarized, since it cannot be extinguished by rotating the analyser.

Experiment II. The arrangements are the same as in the last experiment, and the analyser is turned from the position of extinction, through a small angle towards the right hand of an observer, who is looking through it at the point of incidence, giving a faint restoration of light. When the circuit is closed, so that the reflector becomes a negative pole, the intensity is increased; but when the current is reversed, so that the reflector becomes a positive pole, the intensity is diminished. The weakening effect of the second operation is always less than the strengthening effect of the first, and its effect diminishes as the angle through which the analyser is turned is diminished.

In these two experiments the angle of incidence lay between 60° and 80°; and Kerr does not appear to have observed the effect produced, when the angle of incidence lay between 80° and 90°. See § 463.

453. We must now describe the arrangements, when the incidence is perpendicular.



Instead of employing a wedge for the submagnet, Kerr substituted a block of soft iron *BB*, rounded at one end into the frustum of a cone. A small boring was drilled through the block, narrowing towards the conical ends, and the block was placed next the magnet *M*. The surface of the boring was well coated with lampblack. Above the block, a thin sheet of glass *C* was placed at an angle of 45° to the horizon, which received the horizontal beam from the first Nicol *N*, and reflected it downwards through the boring, perpendicularly to the surface of the reflector. The reflected beam then proceeded back again through the thin sheet of glass and the second Nicol *N'*, which served as the analyser.

454. *Experiment III.* The polarizer and analyser are first placed in the position of extinction, and the analyser is then turned through a small angle, towards the right hand of a person who is looking through it at the point of incidence, giving a faint restoration of light. The circuit is now closed; and it is found, that when the reflector is negatively magnetized, the intensity is increased; and that when it is positively magnetized, the intensity is diminished.

When the analyser is turned towards the left, the results are the same, provided the operations of positive and negative magnetization are reversed.

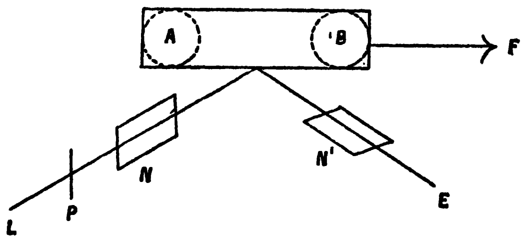
455. As the result of his experiments, Kerr deduced the following general law, viz.;—

When plane polarized light is reflected from the pole of an electromagnet, the plane of polarization of the reflected light is turned through a sensible angle, in a direction contrary to that of the amperean current, which would produce the magnetic force; so that a positive pole of polished iron acting as a reflector, turns the plane of polarization towards the right hand of an observer, looking at the point of incidence along the reflected ray.

456. In the preceding experiments, the reflector was supposed to be magnetized perpendicularly to its surface. We shall now describe the second series of experiments made by Kerr, for the purpose of investigating the effect of a reflector, which is magnetized parallel to its surface[1].

[1] *Phil. Mag.* March, 1878.

457. The electromagnet stands upright upon a table, and a rectangular prism of soft iron, one of whose faces is carefully planed and polished, lies upon the poles of the magnet, with its polished face vertical. The two Nicols N, N', and the lamp L, stand upon the same table as the magnet, and at the same height as the mirror.



The arrangement is shown in the figure; AB is the reflector, E is the eye of the observer, and the dotted lines represent the poles of the magnet. P is a metallic screen, containing a slit $\frac{1}{8}$th of an inch wide, placed between the first Nicol and the lamp.

In the above arrangement, the magnetic force is very nearly parallel to the reflector, and may be conceived to be produced by currents circulating spirally round the prism AB from one pole to the other. Such a current will be considered right-handed, when its direction is towards the right hand of an observer viewing it from F; and a rotation of the analyser N', which is in the direction of the hands of a watch, when viewed from E, will be considered right-handed.

458. In the following two experiments the plane of incidence is parallel to the direction of magnetization.

Experiment IV. The incident light is polarized *in* the plane of incidence, and the analyser is initially placed in the position of extinction, and is then turned through a small angle. The circuit is now closed; and it is found, that the light restored from extinction by a small right-handed rotation of the analyser, is always strengthened by a right-handed magnetizing current, and always weakened by a left-handed current. Conversely the light restored by a small left-handed rotation, is always weakened by a right-handed current, and strengthened by a left-handed one.

The intensity of these optical effects of magnetization varies with the angle of incidence. At an incidence of 85° the effects

are very faint; at 75° they are stronger; at incidences from 65° to 60°, they are clear and strong; at 45° they are fairly strong, though fainter than at 60°; at 30° they are again very faint, and much the same as at 85°.

Experiment V. The incident light is polarized *perpendicularly* to the plane of incidence, and the arrangements are the same as in the last experiment. At an incidence of 85°, the light restored by a right-handed rotation of the analyser is strengthened by a right-handed current, and weakened by a left-handed one; and the effects are undistinguishable from those of the fourth experiment, except that they are considerably weaker. At 80°, the effects are of the same kind, but a good deal fainter; and at 75° they disappear. At 70°, they reappear faintly, but the phenomena are now of a contrary character; for the light restored by a right-handed rotation, is now weakened by a right-handed current, and strengthened by a left-handed one. At incidences of 65°, 60°, 45°, 30°, the effects are of the same kind as at 70°; and at 60° they are comparatively clear and strong, though sensibly fainter than those obtained in the last experiment at the same incidence. At 30° they are faint, but stronger than the contrary effects obtained at 85°.

459. The results of the last two experiments may be summed up as follows:—

(i) *When the incident light is polarized in the plane of incidence, the plane of polarization of the reflected light is always rotated in the opposite direction to that of the amperean current, which would produce the magnetic force.*

(ii) *When the incident light is polarized perpendicularly to the plane of incidence, the rotation of the plane of polarization of the reflected light is in the opposite direction to that of the current, so long as the angle of incidence lies between* 90° *and* 75°; *and in the same direction, when it lies between* 75° *and* 0°.

460. *Experiment VI.* In this experiment, the plane of incidence was perpendicular to the direction of magnetization; and it was found that no optical effect was produced by magnetization.

Experiment VII. In this experiment, the incidence was normal, and the inclination of the plane of polarization to the direction of magnetization was varied from 0° to 90°; and it was found, that no optical effect was produced by magnetization.

461. Dr E. H. Hall[1] of Baltimore has examined the effects produced, when the electromagnet is composed of nickel and of cobalt; and he found that in both metals, the sign of Kerr's effect was the same as in iron.

462. The experiments of Kerr would lead us to anticipate, that when light is reflected from a conductor, which is strongly charged with electricity, the reflected light would experience certain modifications; but no experiments of this character appear to have been performed.

Kundt's Experiments.

463. The experiments of Kerr were repeated by Kundt[2], and were completely confirmed by the latter with one exception, viz. that when light polarized *perpendicularly* to the plane of incidence is reflected at the pole of an iron electromagnet, the direction of rotation is reversed at an incidence of about 82°; that is to say, the rotation is in the *contrary* direction to that of the current so long as the angle of incidence lies between 0° and 82°, and in the *same* direction when it lies between 82° and 90°.

464. When light was reflected from the pole of a nickel electromagnet, it was found that the rotation was more feeble than that produced by iron. When the light was polarized in the plane of incidence, the rotation was always negative (that is in the contrary direction to that of the current); but when the light was polarized perpendicularly to the plane of incidence, the rotation was negative from 0° to 50°, and changed sign between 50° and 60°.

465. Kundt also made experiments upon the rotation produced, when light is transmitted through films of iron, cobalt and nickel, which were so thin as to be semi-transparent; and he found, that all these metals, when magnetized perpendicularly to the surface of the film, produced a powerful rotation of the plane of polarization of the transmitted light; and that the rotation takes place *in* the direction of the magnetizing current. The rotation produced by iron upon the mean rays of the spectrum, is

[1] *Phil. Mag.* Sep. 1881, p. 171.

[2] *Berlin Sitzungsberichte*, July 10th, 1884; translated *Phil. Mag.* Oct. 1884, p. 308.

more than 30,000 times as great as that produced by glass of equal thickness; that produced by cobalt is nearly the same; whilst that produced by nickel is decidedly weaker, being only about 14,000 times greater than that produced by glass.

466. All these metals exhibited rotatory dispersion. The dispersion produced by cobalt and nickel was feeble, whilst that produced by iron was much more powerful, and was *anomalous*; for iron was found to rotate red light to a greater extent than blue.

467. Kundt also made the following experiments upon magnetized glass, which are of some importance, inasmuch as they afford an experimental test of the theory, which will afterwards be proposed.

The poles of a large electromagnet were adjusted at a distance of about 3 cms. apart. A glass plate, the sides of which were not quite accurately parallel, so that the rays reflected from the posterior surface were well separated from those reflected at the anterior surface, was laid upon the poles of an electromagnet. The lines of magnetic force were accordingly parallel to the reflecting surface; also the plane of incidence was parallel to the lines of magnetic force, and the polarizing angle of the glass was $56^{\circ}\cdot4'$.

The light which had been twice refracted at the anterior surface and once reflected at the posterior surface, was examined on emergence; and it was found, that when the incident light was polarized in the plane of incidence, the plane of polarization of the emergent light was always rotated in the *positive* direction; but that when the light was polarized perpendicularly to the plane of incidence, the rotation was negative from normal incidence up to the polarizing angle, and positive from the polarizing angle to grazing incidence.

When the glass plate was magnetized perpendicularly to the reflecting surface, it was found that when the incident light was polarized in the plane of incidence, the rotation was always positive; but that when it was polarized perpendicularly to the plane of incidence, the rotation was positive from normal incidence to the polarizing angle, and negative from the polarizing angle to grazing incidence.

It thus appears, that with regard to the reflected light, the glass plate behaves in an opposite manner to that of iron, nickel and cobalt. With respect however to the transmitted light, glass behaves in the same manner as these metals.

468. Kundt sums up the facts connected with the electromagnetic rotation of the plane of polarization of light as follows.

(i) Most isotropic solid bodies, fluids and those gases, which have been examined, rotate the plane of polarization of the transmitted light in the positive direction.

(ii) A concentrated solution of perchloride of iron produces a negative rotation.

(iii) Oxygen, which is comparatively powerfully magnetic, produces positive rotation.

(iv) When light is transmitted through a thin film of iron, cobalt or nickel, the rotation is positive.

(v) When light is reflected at normal incidence from a magnetic pole of iron, cobalt or nickel, the rotation is negative.

(vi) Upon passing through, as well as upon reflection from, iron, the rotatory dispersion of the light is anomalous; the red rays being rotated more powerfully than the blue.

Hall's Effect.

469. Before we proceed to the theoretical explanation of these phenomena, we must refer to a very important experimental fact, which was discovered by Dr E. H. Hall[1] of Baltimore. He found that, *when an electric current passes through a conductor, which is placed in a strong field of magnetic force, an electromotive force is produced, whose intensity is proportional to the product of the current and the magnetic force, and whose direction is at right angles to the plane containing the current and the magnetic force.*

Hence if α, β, γ be the components of the external magnetic force, u, v, w those of the current, and P', Q', R', those of the additional electromotive force, we shall have

$$P' = -C(\gamma v - \beta w),\quad Q' = -C(\alpha w - \gamma u),\quad R' = -C(\beta u - \alpha v)\ldots(1).$$

[1] *Phil. Mag.* March, 1880.

470. The constant C is a quantity, which depends upon the physical constitution of the medium through which the current is flowing. We shall refer to it as *Hall's constant*, and to the additional electromotive force as *Hall's effect*.

471. Let the conductor consist of a plane plate, which will be chosen as the plane of xy; let the magnetic force be in the positive direction of the axis of z, and let the primary current flow along the positive direction of the axis of y. Then

$$P' = -C\gamma v, \quad Q' = 0, \quad R' = 0.$$

The additional electromotive force will therefore act in the positive or negative direction of the axis of x, according as C is negative or positive. We may express this by saying, that Hall's effect is positive, when Hall's constant is negative.

472. Various experiments have been made for the purpose of determining the magnitude and sign of Hall's effect, the description of which more properly belongs to a treatise on Electromagnetism than to one on Optics[1]. It will however be desirable to call attention to the experiments of Von Ettinghausen and Nernst[2], who found the following values for Hall's effect, its value for tin being taken as unity.

Copper	−13	Nickel	−605
Silver	−21	Antimony	+4800
Gold	−28	Carbon	−4400
Cobalt	+115	Bismuth	−252,5000
Iron	+285	Tellurium	+13,250,000

They found in addition, that the effect was positive in steel, lead, zinc, and cadmium; but negative in all the other metals which they examined.

[1] *Phil. Mag.* Sept. 1881, p. 157; *Ibid.* (5) XVII. pp. 80, 249, 400.
[2] *Amer. Journ. of Science*, (3rd Series), XXXIV. p. 151; and *Nature*, 1887, p. 185.

Theory of Magnetic Action on Light.

473. In the experiments of Kerr and Kundt, on reflection from magnets, and transmission through thin magnetized films, the substance experimented upon was a metal. It is therefore hopeless to attempt to construct a theory, which will furnish a complete explanation of these phenomena, until a satisfactory theory of metallic reflection has been obtained. The theory of magnetic action on light, which we shall now consider[1], only applies to transparent media, and depends upon the experimental result discovered by Hall, which has been discussed in the preceding sections.

Now Professor Rowland[2] has assumed, that this result holds good in a dielectric, which is under the action of a strong magnetic force; if, therefore, we adopt this hypothesis, we must substitute the time variations of the electric displacement for the current, and the equations of electromotive force become

$$\left.\begin{aligned} P &= -\frac{dF}{dt} - C(\gamma\dot{g} - \beta\dot{h}) - \frac{d\psi}{dx} \\ Q &= -\frac{dG}{dt} - C(\alpha\dot{h} - \gamma\dot{f}) - \frac{d\psi}{dy} \\ R &= -\frac{dH}{dt} - C(\beta\dot{f} - \alpha\dot{g}) - \frac{d\psi}{dz} \end{aligned}\right\} \quad \text{.............. (1)},$$

where α, β, γ are the components of the total magnetic force.

When the magnetic field is disturbed by the passage of a wave of light, α, β, γ may be supposed to have the same values as before disturbance, since their variations when multiplied by $\dot{f}$, $\dot{g}$, $\dot{h}$ are terms of the second order, which may be neglected. Since we shall confine our attention to the propagation of light in a uniform magnetic field, α, β, γ may be regarded as constant quantities.

We shall therefore assume, that when light is transmitted through a medium, which, when under the action of a strong magnetic force, is capable of magnetically affecting light, the

[1] *Phil. Trans.* 1891, p. 371. For other theories, see Maxwell, *Electricity and Magnetism*, vol. II. ch. xxi.; Fitzgerald, *Phil. Trans.* 1880, p. 691.

[2] *Phil. Mag.*, April, 1881, p. 254.

equations of electromotive force are represented by (1), where C is Hall's constant. Since we shall require to use the letters α, β, γ to denote that portion of the magnetic force which is due to optical causes, we shall write these equations in the form

$$\left.\begin{aligned} P &= -\frac{dF}{dt} - p_3\dot{g} + p_2\dot{h} - \frac{d\psi}{dx} \\ Q &= -\frac{dG}{dt} - p_1\dot{h} + p_3\dot{f} - \frac{d\psi}{dy} \\ R &= -\frac{dH}{dt} - p_2\dot{f} + p_2\dot{g} - \frac{d\psi}{dz} \end{aligned}\right\} \qquad (2),$$

where $p_1 = C\alpha$, &c.

All the other equations of the field are the same as Maxwell's, with the exception that we do not suppose that

$$dF/dx + dG/dy + dH/dz,$$

is zero.

474. In order to obtain the equations of electric displacement, let us consider a medium which is magnetically isotropic but electrostatically æolotropic. Let k be the magnetic permeability; K_1, K_2, K_3 the three principal electrostatic capacities; also let

$$kK_1 = A^{-2}, \quad kK_2 = B^{-2}, \quad kK_3 = C^{-2}.$$

$$\Omega = A^2 \frac{df}{dx} + B^2 \frac{dg}{dy} + C^2 \frac{dh}{dz} \qquad (3),$$

$$\frac{d}{d\omega} = p_1 \frac{d}{dx} + p_2 \frac{d}{dy} + p_3 \frac{d}{dz} \qquad (4).$$

From the last two of (2) we obtain

$$\frac{da}{dt} + p_1 \frac{d\dot{f}}{dy} + p_3 \frac{d\dot{f}}{dz} - p_1 \frac{d\dot{g}}{dy} - p_1 \frac{d\dot{h}}{dz} = \frac{dQ}{dz} - \frac{dR}{dy}.$$

Substituting the values of P, Q, R from the equations $P = 4\pi f/K_1$, &c., and recollecting that

$$\frac{df}{dx} + \frac{dg}{dy} + \frac{dh}{dz} = 0 \qquad (5),$$

we obtain $$\frac{da}{dt} = 4\pi k \left(B^2 \frac{dg}{dz} - C^2 \frac{dh}{dy}\right) - \frac{d\dot{f}}{d\omega} \qquad (6),$$

with two similar equations.

Now $$4\pi k\ddot{f} = 4\pi k\dot{u} = \frac{d}{dt}\left(\frac{dc}{dy} - \frac{db}{dz}\right) \quad\text{..............(7)},$$

substituting the values of $\dot{a}$, $\dot{b}$, $\dot{c}$ from (6) we obtain

$$\left.\begin{aligned}\frac{d^2f}{dt^2} &= A^2\nabla^2 f - \frac{d\Omega}{dx} + \frac{1}{4\pi k}\frac{d}{d\omega}\left(\frac{d\dot{g}}{dz} - \frac{d\dot{h}}{dy}\right)\\ \frac{d^2g}{dt^2} &= B^2\nabla^2 g - \frac{d\Omega}{dy} + \frac{1}{4\pi k}\frac{d}{d\omega}\left(\frac{d\dot{h}}{dx} - \frac{d\dot{f}}{dz}\right)\\ \frac{d^2h}{dt^2} &= C^2\nabla^2 h - \frac{d\Omega}{dz} + \frac{1}{4\pi k}\frac{d}{d\omega}\left(\frac{d\dot{f}}{dy} - \frac{d\dot{g}}{dx}\right)\end{aligned}\right\}\quad\text{.........(8).}$$

These are the equations satisfied by the components of electric displacement.

475. We shall now confine our attention to isotropic media. In this case $A = B = C = U$, where $U^{-2} = kK$; hence (6) becomes

$$\frac{da}{dt} = 4\pi k U^2\left(\frac{dg}{dz} - \frac{dh}{dy}\right) - \frac{d\dot{f}}{d\omega}.$$

Let $f = S\lambda$, $g = S\mu$, $h = S\nu$, $S = e^{2\iota\pi/V\tau.(lx+my+nz-Vt)}$,

then $$\frac{da}{dt} = \frac{8\iota\pi^2U^2k}{V\tau}(n\mu - m\nu)S - \frac{2\iota\pi}{V\tau}(lp_1 + mp_2 + np_3)\dot{S}\lambda,$$

whence

$$\alpha = -\frac{4\pi U^2}{V}(n\mu - m\nu)S - \frac{2\iota\pi}{Vk\tau}(lp_1 + mp_2 + np_3)S\lambda.$$

Accordingly if $\mathfrak{H}$ denote the component of the external magnetic force perpendicular to the wave-front, the equations of magnetic force become

$$\left.\begin{aligned}\alpha &= -\frac{4\pi U^2}{V}(ng - mh) - \frac{2\iota\pi}{Vk\tau}C\mathfrak{H}f\\ \beta &= -\frac{4\pi U^2}{V}(lh - nf) - \frac{2\iota\pi}{Vk\tau}C\mathfrak{H}g\\ \gamma &= -\frac{4\pi U^2}{V}(mf - lg) - \frac{2\iota\pi}{Vk\tau}C\mathfrak{H}h\end{aligned}\right\}\quad\text{.........(9),}$$

where C is Hall's constant.

Propagation of Light.

476. We are now prepared to consider the propagation of light in a magnetized medium.

Let us suppose that plane waves of light are incident upon the surface of separation of air and a magnetized medium. Let the axis of x be the normal, and be drawn into the first medium, and let the axis of z be perpendicular to the plane of incidence; also let the direction of magnetization be parallel to the axis of x. Then $p_2 = p_3 = 0$, and none of the quantities are functions of z; whence the equations of motion become

$$\left.\begin{aligned} \frac{d^2 f}{dt^2} &= U^2\nabla^2 f - \frac{p}{4\pi k}\frac{d^2 h}{dx\,dy} \\ \frac{d^2 g}{dt^2} &= U^2\nabla^2 g + \frac{p}{4\pi k}\frac{d^2 h}{dx^2} \\ \frac{d^2 h}{dt^2} &= U^2\nabla^2 h + \frac{p}{4\pi k}\frac{d}{dx}\left(\frac{df}{dy} - \frac{dg}{dx}\right) \end{aligned}\right\} \quad\text{............(10).}$$

where p is written for p_1.

Let

$$f = A'S, \quad g = A''S, \quad h = AS, \quad S = \epsilon^{2\iota\pi/V\tau\cdot(lx+my-Vt)}.$$

Substituting in (10), we obtain

$$(U^2 - V^2)A' = \frac{\iota p}{2k\tau}\,lmA,$$

$$(U^2 - V^2)A'' = \frac{\iota p}{2k\tau}\,l^2 A,$$

$$(U^2 - V^2)A \; = \frac{\iota pl}{2k\tau}(A'm - A''l).$$

From these equations we deduce

$$V^2 = U^2 \pm \frac{pl}{2k\tau} \quad\text{.......................(11),}$$

whence $\qquad A' = \pm\,\iota mA, \quad A'' = \mp\,\iota lA.$

Hence, if V_1, V_2 denote the two values of V corresponding to the upper and lower signs, we see that two waves are propagated with velocities V_1, V_2.

It is important to notice, that the directions of the two refracted waves corresponding to an incident wave are in general different.

To see this, let the suffixes 1 and 2 refer to the two refracted waves, and let the incident wave be

$$h = \epsilon^{2\iota\pi/V\tau \cdot (lx+my-Vt)},$$

then the displacements in one of the refracted waves will be

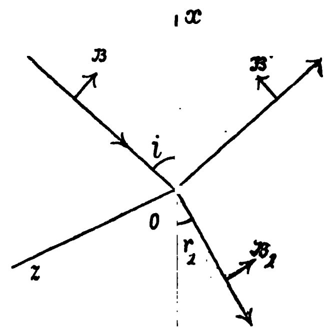
$$f_1 = \iota m_1 A_1 S_1, \quad g_1 = -\iota l_1 A_1 S_1, \quad h_1 = A_1 S_1,$$

where $$S_1 = \epsilon^{2\iota\pi/V_1\tau \cdot (l_1x+m_1y-V_1t)},$$

and the displacements in the other wave will be obtained by changing the suffix from 1 to 2, and changing the signs of f_1, g_1. Now, if r_1, r_2 be the angles of refraction, $m_1 = \sin r_1$, $m_2 = \sin r_2$; and, since the coefficient of y must be the same in all three waves, we must have

$$\frac{V}{\sin i} = \frac{V_1}{\sin r_1} = \frac{V_2}{\sin r_2},$$

which shows that r_1 is different from r_2.



Let $\mathfrak{B}_1$, $\mathfrak{B}_2$ be the component displacements in the plane $z=0$, then since $l_1 = -\cos r_1$, it follows that

$$\mathfrak{B}_1 = f_1 \sin r_1 + g_1 \cos r_1 = \iota A_1 S_1.$$

Similarly $$\mathfrak{B}_2 = -\iota A_2 S_2.$$

The component displacements perpendicular to the wave-fronts are evidently zero; whence, in real quantities, the displacements in the two waves are

$$h_1 = A_1 \cos \frac{2\pi}{V_1\tau}(l_1x + m_1y - V_1t),$$

$$\mathfrak{B}_1 = -A_1 \sin \frac{2\pi}{V_1\tau}(l_1x + m_1y - V_1t),$$

and $$h_2 = A_2 \cos \frac{2\pi}{V_2\tau}(l_2x + m_2y - V_2t),$$

$$\mathfrak{B}_2 = A_2 \sin \frac{2\pi}{V_2\tau}(l_2x + m_2y - V_2t),$$

and, consequently, the two waves are circularly polarized in opposite directions.

477. The results of the last article will enable us to explain the rotation of the plane polarization, when light is propagated through a magnetic field parallel to the direction of the lines of magnetic force. In this case

$$l_1 = l_2 = -1, \quad m_1 = m_2 = 0,$$

whence putting $k = 1$, since the field is a transparent dielectric, we obtain from (11),

$$V_1^2 = U^2 - p/2\tau,$$

$$V_2^2 = U^2 + p/2\tau,$$

accordingly if the waves are travelling along the negative direction of the axis of x,

$$h_1 = A_1 \cos \frac{2\pi}{\tau}\left(\frac{x}{V_1} + t\right), \quad g_1 = \mathfrak{B}_2 = A_2 \sin \frac{2\pi}{\tau}\left(\frac{x}{V_2} + t\right),$$

$$h_2 = A_2 \cos \frac{2\pi}{\tau}\left(\frac{x}{V_2} + t\right), \quad g_2 = \mathfrak{B}_2 = -A_2 \sin \frac{2\pi}{\tau}\left(\frac{x}{V_2} + t\right).$$

We shall hereafter show, that the amplitudes are not quite equal to one another, but are of the form $P + Q$ and $P - Q$ respectively, where Q is a quantity which depends upon the magnetic force. Since the magnetic effect is very small in transparent dielectrics, we may as a first approximation neglect the difference between A_1 and A_2, whence dropping the suffixes, the vibrations in question become

$$g = A \sin \frac{2\pi}{\tau}\left(\frac{x}{V_1} + t\right) - A \sin \frac{2\pi}{\tau}\left(\frac{x}{V_2} + t\right),$$

$$= 2A \cos \frac{2\pi}{\tau}\left\{\tfrac{1}{2}x\left(\frac{1}{V_1} + \frac{1}{V_2}\right) + t\right\} \sin \frac{\pi x}{\tau}\left(\frac{1}{V_1} - \frac{1}{V_2}\right),$$

$$h = 2A \cos \frac{2\pi}{\tau}\left\{\tfrac{1}{2}x\left(\frac{1}{V_1} + \frac{1}{V_2}\right) + t\right\} \cos \frac{\pi x}{\tau}\left(\frac{1}{V_1} - \frac{1}{V_2}\right).$$

Whence if ψ be the angle through which the plane of polarization is rotated, *measured towards the right hand of an observer who is looking along the direction of propagation of the ray,*

$$\tan \psi = -g/h = \tan \frac{\pi x}{\tau}\left(\frac{1}{V_2} - \frac{1}{V_1}\right).$$

Expanding V_1^{-1} and V_2^{-1} in powers of p, and putting $p = C\alpha$, we obtain

$$\frac{1}{V_2} - \frac{1}{V_1} = -\frac{C\alpha}{2U^3\tau},$$

accordingly $$\psi = -\pi x C\alpha / 2U^3\tau^2 \quad\quad (12).$$

Since the wave is travelling in the negative direction of the axis of x, it follows that, if T be the thickness of the medium traversed by the wave, $x = -T$; whence (12) becomes

$$\psi = \pi T C\alpha / 2U^3\tau^2 \quad\quad (13),$$

which shows that the plane of polarization of the emergent light is rotated, and that the direction of rotation depends upon that of the magnetic force.

478. It appears from Faraday's experiments, that the direction of rotation is the same as that of the amperean current, which would produce the magnetic force. Now α is measured along the positive direction of the axis of x, whence the amperean current circulates from the right hand to the left hand of an observer who is looking along the direction of propagation; accordingly C must be negative for glass, whilst for a medium such as perchloride of iron C must be positive.

From these results we draw the following conclusions.

(i) The magnitude of the rotation is directly proportional to the magnetic force, and also to the thickness of the medium traversed; and it is inversely proportional to the square of the period of the light. Hence the rotation is greater for violet light than for red light.

(ii) The direction of rotation is the same as that of the amperean current which would produce the magnetic force, for media for which Hall's constant is negative; and in the opposite direction for media for which Hall's constant is positive.

(iii) When the direction of propagation is perpendicular to that of the magnetic force, it follows from (8), that the magnetic terms are zero; hence the magnetic force produces no optical effect. These results are in accordance with experiment; subject to the limitation, that the effect of rotatory dispersion is only approximately expressed by the first statement.

The Boundary Conditions.

479. When light is reflected and refracted at the surface of separation of two isotropic or crystalline media, the boundary conditions are, (i) that the components of the electromotive and magnetic forces *parallel* to the surface of separation must be continuous; (ii) that the components of electric displacement and magnetic induction *perpendicular* to the surface of separation must likewise be continuous. We have, therefore, *six* equations to determine *four* unknown quantities; but inasmuch as two pairs of these equations are identical, the total number reduces to four, which is just sufficient to determine the four unknown quantities. If, however, we were to assume these six conditions in the case of a magnetized medium, we should find that we should be led to inconsistent results, and we shall, therefore, proceed to prove the boundary conditions.

Since the electric displacement and the magnetic induction both satisfy the equation

$$\frac{df}{dx}+\frac{dg}{dy}+\frac{dh}{dz}=0,$$

which is an equation of the same form as the equation of continuity of an incompressible fluid in Hydrodynamics; it follows that the components of the electric displacement and magnetic induction perpendicular to the surface of separation must be continuous.

To obtain the other conditions, let us suppose, as before, that the plane $x=0$ is the surface of separation, and that the plane $z=0$ contains the direction of propagation. Then, since the coefficients of y and t in the exponential factor must be the same in all four waves, d/dy and d/dt of any continuous function will also be continuous, and conversely. Since none of the quantities are functions of z,

$$4\pi\dot{f}=\frac{d\gamma}{dy},$$

which shows that γ is continuous.

Since the continuity of γ follows from that of f, the conditions of continuity of both these quantities will be expressed by the same equation.

Since $$a=\frac{dH}{dy},$$

it follows that H is continuous, whence if the accents refer to the second medium, we obtain from (2)

$$R' + p_2\dot{f}' - p_1\dot{g}' = R \qquad (14).$$

This equation shows that the electromotive force parallel to z is discontinuous. This circumstance may, at first sight, appear somewhat strange, and may perhaps be regarded as an objection to the theory; but since the p's are exceedingly small quantities, the discontinuity is also very small. We have, moreover, assumed that the transition from one medium to the other is *abrupt*, whereas, if we were better acquainted with the conditions at the confines of two different media, we should probably find that this was not the case; but that there would be a rapid but continuous change in the component of the electromotive force parallel to the boundary, in passing from one medium to the other.

We have, therefore, as yet, only obtained two independent boundary equations. Now, we shall presently see that when plane polarized light is reflected and refracted at the surface of a magnetized medium, the reflected light is elliptically polarized; whilst, as we have already shown, the two refracted waves are circularly polarized in opposite directions. We have, therefore, four unknown quantities to determine, viz., the amplitudes of the two components of the reflected vibration, and the amplitudes of the two refracted waves. We, therefore, require two more equations. To find a third equation, we shall assume, that the component of magnetic force parallel to the axis of y is continuous. A fourth equation will be obtained from the condition of continuity of energy; for since there is no conversion of energy into heat, or any form of energy other than the electrical kind, it follows that the rate of increase of the electrostatic and electrokinetic energies within any closed surface must be equal to the rate at which energy flows in across the boundary.

480. We must now obtain an expression for the energy.

It is a general principle of Dynamics, that if equations are given which are sufficient to completely determine the motion of a system, the Principle of Energy can be deduced from these equations. The proper form of the Principle of Energy in the case of a dielectric medium is this :—*Describe any closed surface in the medium, then the rate at which energy increases within the surface, is equal to the rate at which energy flows in across the boundary.*

If E be the electric energy per unit of volume, the rate at which energy increases within the surface is $\iiint \dot{E}\,dxdydz$, and, consequently, this quantity must be capable of being expressed as a surface integral taken over the boundary; and any form of $\dot{E}$ which is not capable of being so expressed must certainly be wrong. If the medium were a conductor, in which there is a conversion of energy into heat, $\iiint \dot{E}\,dxdydz$ would not be expressible in the form of a surface integral[1], but this case need not be considered, since we are dealing with a transparent dielectric.

Since $P = 4\pi f/K_1 = 4\pi k A^2 f$, equations (6) may be written in the form

$$\frac{da}{dt} = \frac{dQ}{dz} - \frac{dR}{dy} - \frac{d\dot{f}}{d\omega}.$$

Multiply this equation and the two corresponding ones by α, β, γ; then add and integrate throughout any closed surface, and we shall obtain

$$\tfrac{1}{2}k\frac{d}{dt}\iiint (\alpha^2 + \beta^2 + \gamma^2)\,dxdydz$$

$$= \iiint \left\{\alpha\left(\frac{dQ}{dz} - \frac{dR}{dy}\right) + \beta\left(\frac{dR}{dx} - \frac{dP}{dz}\right) + \gamma\left(\frac{dP}{dy} - \frac{dQ}{dx}\right)\right\} dxdydz$$

$$- \iiint \left(\alpha\frac{d\dot{f}}{d\omega} + \beta\frac{d\dot{g}}{d\omega} + \gamma\frac{d\dot{h}}{d\omega}\right) dxdydz \ldots(15).$$

Let $$W = 2\pi k \iiint (A^2 f^2 + B^2 g^2 + C^2 h^2)\,dxdydz \quad \ldots(16),$$

then $$\frac{dW}{dt} = 4\pi k \iiint (A^2 f\dot{f} + B^2 g\dot{g} + C^2 h\dot{h})\,dxdydz$$

$$= \iiint (P\dot{f} + Q\dot{g} + R\dot{h})\,dxdydz.$$

Substituting the values of $\dot{f}$, $\dot{g}$, $\dot{h}$ in terms of α, β, γ, and integrating by parts, we obtain

$$\frac{dW}{dt} = \frac{1}{4\pi}\iint \{l(R\beta - Q\gamma) + m(P\gamma - R\alpha) + n(Q\alpha - P\beta)\}\,dS$$

$$-\frac{1}{4\pi}\iiint \left\{\alpha\left(\frac{dQ}{dz} - \frac{dR}{dy}\right) + \beta\left(\frac{dR}{dx} - \frac{dP}{dz}\right) + \gamma\left(\frac{dP}{dy} - \frac{dQ}{dx}\right)\right\} dxdydz$$

$$\ldots\ldots\ldots(17).$$

[1] See Poynting, *Phil. Trans.*, 1884, p. 343.

If in the identity

$$\dot{f}(p_3\dot{g}-p_2\dot{h})+\dot{g}(p_1\dot{h}-p_3\dot{f})+\dot{h}(p_2\dot{f}-p_1\dot{g})=0,$$

we substitute the values of $\dot{f}$, $\dot{g}$, $\dot{h}$ from the equations of

$$4\pi\dot{f}=d\gamma/dy-d\beta/dz, \text{ \&c.},$$

in the coefficients of the terms in brackets, and integrate by parts, we shall find that the last volume integral in (15) is equal to

$$-\iint [l\{(p_2\dot{f}-p_1\dot{g})\beta-(p_1\dot{h}-p_3\dot{f})\gamma\}$$
$$+m\{(p_3\dot{g}-p_2\dot{h})\gamma-(p_2\dot{f}-p_1\dot{g})\alpha\}$$
$$+n\{(p_1\dot{h}-p_3\dot{f})\alpha-(p_3\dot{g}-p_2\dot{h})\beta\}]\,dS \quad \text{...(18).}$$

Accordingly (15) becomes on substitution from (16), (17) and (18)

$$\frac{d}{dt}\iiint\left\{\frac{k}{8\pi}(\alpha^2+\beta^2+\gamma^2)+2\pi k(A^2f^2+B^2g^2+C^2h^2)\right\}dxdydz$$
$$=\frac{1}{4\pi}\iint[l\{(R+p_2\dot{f}-p_1\dot{g})\beta-(Q+p_1\dot{h}-p_3\dot{f})\gamma\}$$
$$+m\{(P+p_3\dot{g}-p_2\dot{h})\gamma-(R+p_2\dot{f}-p_1\dot{g})\alpha\}$$
$$+n\{(Q+p_1\dot{h}-p_3\dot{f})\alpha-(P+p_2g-p_2\dot{h})\beta\}]\,dS \quad \text{...(19).}$$

The physical interpretation of this equation is, that the rate at which something increases within the closed surface must be equal to the rate at which something flows into the surface. This cannot be anything else but energy; we are therefore led to identify the expression

$$\frac{k}{8\pi}(\alpha^2+\beta^2+\gamma^2)+2\pi k(A^2f^2+B^2g^2+C^2h^2),$$

as representing the energy of the electric field per unit of volume. The first term represents the electrokinetic energy, and the second term the electrostatic energy.

The above expressions are the same as those obtained by Maxwell by a different method, and it thus appears that the expressions for each species of energy are not altered by the additional terms, which have been introduced into the general equations of electromotive force.

The right-hand side of (19) represents the rate at which work is done by the electric and magnetic forces, which act upon the surface of S.

481. In the optical problem which we are considering, the bounding surface is the plane $x=0$; whence if the quantities in the magnetized medium be denoted by accented letters, the condition of continuity of energy becomes

$$R\beta - Q\gamma = (R' + p_2\dot{f}' - p_1\dot{g}')\beta' - (Q' + p_1\dot{h}' - p_3\dot{f}')\gamma'.$$

Since $\beta = \beta'$ and $\gamma = \gamma'$, it follows from (14) that this equation reduces to

$$Q = Q' + p_1\dot{h}' - p_3\dot{f}' \quad \text{.....................(20)},$$

which shows that the component of the electromotive force *in* the plane of incidence is also discontinuous.

The boundary conditions are therefore the following; (i) continuity of electric displacement perpendicular to the reflecting surface, which is equivalent to continuity of magnetic force parallel to z; (ii) continuity of magnetic induction perpendicular to the reflecting surface, which is also equivalent to equation (14); (iii) continuity of magnetic force parallel to y; (iv) equation (20), which follows partly from (i), (ii) and (iii), and partly from the condition that the flow of energy must be continuous.

We have therefore four equations, and no more, to determine the four unknown quantities.

Reflection and Refraction.

482. We shall now calculate the amplitudes of the reflected and refracted waves, when light is reflected and refracted at the surface of a transparent medium which is magnetized normally, so that $p_2 = p_3 = 0$.

Let A, B be the amplitudes of the two components of the incident light perpendicular to, and in the plane of incidence; then the displacements in the four waves may be written

$$h = AS, \qquad \mathfrak{H} = BS, \qquad \text{incident wave;}$$
$$h' = A'S', \qquad \mathfrak{H}' = B'S', \qquad \text{reflected wave;}$$
$$h_1 = A_1S_1, \qquad \mathfrak{H}_1 = \iota A_1S_1, \qquad \text{1st refracted wave;}$$
$$h_2 = A_2S_2, \qquad \mathfrak{H}_2 = -\iota A_2S_2, \qquad \text{2nd refracted wave.}$$

Also

$$l = -\cos i, \quad l' = \cos i, \quad l_1 = -\cos r_1, \quad l_2 = -\cos r_2,$$
$$m = \sin i, \quad m' = \sin i, \quad m_1 = \sin r_1, \quad m_2 = \sin r_2.$$

The boundary conditions (i), (ii), (iii), (iv), of § 481, furnish the following equations :

$$(B+B')V=\iota(A_1V_1-A_2V_2)\ldots\ldots\ldots\ldots\ldots(21),$$

$$(A+A')V^2=U^2k(A_1+A_2)-\frac{p}{2\tau}(A_1\cos r_1-A_2\cos r_2)\ (22),$$

$$(A-A')V\cos i=U^2\left(\frac{A_1}{V_1}\cos r_1+\frac{A_2}{V_2}\cos r_2\right.$$
$$\left.-\frac{p}{2k\tau}\left(\frac{A_1}{V_1}\cos^2 r_1-\frac{A_2}{V_2}\cos^2 r_2\right)\right)\ldots\ldots\ldots(23),$$

$$(B-B')V^2\cos i=\iota U^2k(A_1\cos r_1-A_2\cos r_2)-\frac{\iota p}{2\tau}(A_1+A_2)\ (24),$$

where p is written for p_1.

We shall now simplify these equations by introducing an auxiliary angle R, such that

$$\frac{V}{\sin i}=\frac{V_1}{\sin r_1}=\frac{V_2}{\sin r_2}=\frac{U}{\sin R}\ldots\ldots\ldots\ldots\ldots(25).$$

Hence, R is the angle of refraction when the second medium is unmagnetized; and accordingly r_1 and r_2 will differ from R by a small quantity which depends upon p. Since the magnetic effects are small, we shall neglect squares and higher powers of p, and we may, therefore, in the terms multiplied by p, put $r_1=r_2=R$.

Let $q=p/4k\tau$, then from (11)

$$V_1=U-\frac{q\cos R}{U},\quad V_2=U+\frac{q\cos R}{U}\ \ldots\ldots\ldots\ (26),$$

and also from (25) and (26) we obtain

$$\cos r_1=\cos R+\frac{q\sin^2 R}{U^2},\ \cos r_2=\cos R-\frac{q\sin^2 R}{U^2}\ \ldots(27).$$

Substituting these values in equations (21) to (24) and reducing, they finally become

$$\left.\begin{aligned}(B-B')V^2\cos i&=\iota U^2k(A_1-A_2)\cos R+\iota qk(\sin^2 R-2)(A_1+A_2)\\ (B+B')V&=\iota U(A_1-A_2)-\frac{\iota q\cos R}{U}(A_1+A_2)\\ (A-A')V\cos i&=U(A_1+A_2)\cos R-\frac{q\cos 2R}{U}(A_1-A_2)\\ (A+A')V^2&=U^2k(A_1+A_2)-2qk(A_1-A_2)\cos R\end{aligned}\right\}$$
$$\ldots\ldots\ldots(28).$$

These equations determine the amplitudes of the reflected and refracted waves, when the magnetization is perpendicular to the reflecting surface.

From the first two of (28) we get

$$\iota q k (A_1 + A_2) = BV(Uk\cos R - V\cos i) + B'V(Uk\cos R + V\cos i)$$

$$\iota U^2 k (A_1 - A_2) = BV\{Uk(2 - \sin^2 R) - V\cos i\cos R\}$$
$$+ B'V\{Uk(2 - \sin^2 R) + V\cos i\cos R\}.$$

Substituting in the last two, we obtain

$$(A - A')\cos i$$
$$= -\iota B\left[\frac{U\cos R}{qk}(Uk\cos R - V\cos i) - \frac{q\cos 2R}{U^3 k}\{Uk(2-\sin^2 R) - V\cos i\cos R\}\right]$$
$$-\iota B'\left[\frac{U\cos R}{qk}(Uk\cos R + V\cos i) - \frac{q\cos 2R}{U^3 k}\{Uk(2-\sin^2 R) + V\cos i\cos R\}\right] \quad \ldots (29),$$

and

$$(A + A')V$$
$$= -\iota B\left[\frac{U^2}{q}(Uk\cos R - V\cos i) - \frac{2q\cos R}{U^2}\{Uk(2-\sin^2 R) - V\cos i\cos R\}\right]$$
$$-\iota' B\left[\frac{U^2}{q}(Uk\cos R + V\cos i) - \frac{2q\cos R}{U^2}\{Uk(2-\sin^2 R) + V\cos i\cos R\}\right] \quad \ldots (30).$$

Solving these equations, we obtain

$$A' = \frac{A(Uk\cos i - V\cos R)}{Uk\cos i + V\cos R} + \frac{2\iota qkBV\cos i}{U(Uk\cos R + V\cos i)(Uk\cos i + V\cos R)} \quad \ldots\ldots\ldots (31),$$

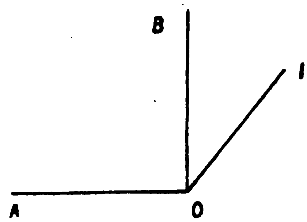
$$B' = -\frac{B(Uk\cos R - V\cos i)}{Uk\cos R + V\cos i} + \frac{2\iota qkAV\cos i}{U(Uk\cos R + V\cos i)(Uk\cos i + V\cos R)} \quad \ldots\ldots\ldots (32).$$

483. We shall now discuss these results.

Equations (31) and (32) give the amplitudes of the two components of the reflected light, and we see that the magnetic terms vanish at grazing incidence, but do not vanish for any other incidence.

The equations may be written in the form

$$\left.\begin{aligned} A' &= A\alpha + \iota q B\beta \\ B' &= B\gamma + \iota q A\beta \end{aligned}\right\} \quad\text{.....................} \quad (33).$$



In the figure let I be the point of incidence, IO the normal to the reflected wave, and let O be the observer; also let OA, OB be drawn at right angles to OI, perpendicular to and in the plane of incidence respectively. Let ξ, η be the displacements along OA, OB; also let $\phi = (2\pi/\lambda)(x \cos i + y \sin i - Vt)$. Then by (33)

$$\left.\begin{aligned} \xi &= A\alpha \cos\phi - qB\beta \sin\phi \\ \eta &= B\gamma \cos\phi - qA\beta \sin\phi \end{aligned}\right\} \quad\text{.................} (34),$$

which shows that the reflected light is elliptically polarized.

Let us first suppose that the incident light is polarized *in* the plane of incidence, so that $B = 0$, and let the principal section of the analyser coincide with OB. Then the intensity of the reflected light after it has passed through the analyser is proportional to $A^2\beta^2q^2$, and is therefore independent of the direction of the magnetizing current, and vanishes when the current is cut off.

Secondly, let the analyser be turned through a small angle ϵ towards the right hand of the observer. From (34) we see that the intensity of the reflected light after emerging from the analyser is proportional to

$$A^2(\alpha^2\epsilon^2 + \beta^2q^2),$$

from which it appears that the effect of the current is always to increase the intensity, and that the intensity is independent of the direction of the current.

The first result is in accordance with the first of Kerr's experiments, but the second is not; since he found under these circumstances, that if a current in one direction strengthened the reflected light, a current in the opposite direction weakened it. We must however recollect, that in Kerr's experiments a polished plate of soft iron was employed, and consequently his results were

affected by the influence of metallic reflection; it is therefore hopeless to attempt to construct a theory which will furnish a theoretical explanation of Kerr's experiments, until a satisfactory electromagnetic theory of metallic reflection has been obtained.

484. When light is reflected or refracted at the surface of a transparent medium, which is magnetized parallel to the reflecting surface, the problem can be worked out in a similar manner to that employed in § 482; but for this the reader is referred to my original paper[1]. It will be found that in this case also, the intensity of the reflected light is independent of the direction of the magnetic force; whereas Kerr's experiments show, that the reverse is the case when the reflector is a metal. When however the plane of incidence is perpendicular to the lines of magnetic force, or when the incidence is normal, magnetization produces no optical effect. This result follows from equations (8), and agrees with the experiments of Faraday, Kerr and Kundt.

485. The experiments of Kundt described in § 467, in which light was incident upon a magnetized plate of glass, furnish a means of subjecting this theory to an experimental test; and we shall therefore consider the case in which the lines of magnetic force are perpendicular to the faces of the plate, and the incidence is sensibly normal, and shall calculate the intensity of the light which has undergone two refractions at the anterior surface, and one reflection at the posterior.

Let the incident vibration be $f=0$, $g=0$, $h=2A\epsilon^{-2\iota\pi t/\tau}$; we shall find it convenient to resolve this into the two circularly polarized waves,

$$h = AS, \qquad g = \iota AS \quad \text{.....................(35)},$$

$$h = AS, \qquad g = -\iota AS \quad \text{.................(36)},$$

where $S = \epsilon^{-2\iota\pi t/\tau}$.

Since we are dealing with glass, we may put $k=1$; consequently for normal incidence we obtain from (31), (32), and (35)

$$A' = A(\alpha - q\beta), \qquad B' = -\iota A'$$

where

$$\alpha = \frac{U-V}{U+V}, \qquad \beta = \frac{2V}{U(U+V)^2} \quad \text{...............(37)}.$$

[1] *Phil. Trans.*, 1891, §§ 12—14.

If, however, the incident wave is polarized in the opposite direction, and is therefore represented by (36), we shall obtain

$$A' = A(\alpha + q\beta), \quad B' = \iota A'$$

in which the sign of q is reversed.

In order to calculate the intensity of the refracted light, let us first confine our attention to the incident wave given by (35). Then if we put $i = R = 0$ in (28), we shall obtain $A_2 = 0$,

$$A_1 = \frac{2AV^2}{U}\left\{\frac{1}{U+V} + \frac{q(2U+V)}{U^2(U+V)^2}\right\} \quad \text{..........(38).}$$

If, however, we considered the other wave (36), we should obtain $A_1 = 0$,

$$A_2 = \frac{2AV^2}{U}\left\{\frac{1}{U+V} - \frac{q(2U+V)}{U^2(U+V)^2}\right\} \quad \text{..........(39),}$$

in which the sign of q is again reversed.

In order to calculate the intensities, when light propagated in glass is reflected at the surface in contact with air, we may use Stokes's Principle of Reversion[1], and apply it separately to each of the two circularly polarized waves, into which the incident wave may be conceived to be resolved. Let Ab, Ac be the amplitudes of the reflected and refracted waves, when the wave (35) passes from air into glass; and let Ae, Af be the amplitudes when the wave passes from glass into air, then

$$b + e = 0, \quad b^2 + cf = 1 \quad \text{..................(40).}$$

Also if we denote by accented letters the corresponding quantities for the other wave (36), the values of b', c', e', f' will be obtained from those of b, c, e, f by writing $-q$ for q.

By (37) and (38), the values of b, c, e, f are

$$\left.\begin{aligned} b &= \frac{U-V}{U+V} - \frac{2qV}{U(U+V)^2} = -e \\ c &= \frac{2V^2}{U}\left\{\frac{1}{U+V} + \frac{q(2U+V)}{U^2(U+V)^2}\right\} \\ f &= \frac{2U^2}{V}\left\{\frac{1}{U+V} - \frac{q(2V+U)}{U^2(U+V)^2}\right\} \end{aligned}\right\} \quad \text{.......... (41),}$$

which can also be verified by independent calculation.

[1] Ante, p. 29.

If r be the thickness of the plate, the emergent light is

$$h = A\left(cef\epsilon^{4\iota\pi r/V_1\tau} + c'e'f'\epsilon^{4\iota\pi r/V_2\tau}\right)\epsilon^{-2\iota\pi t/\tau}$$

$$g = \iota A\cdot\left(cef\epsilon^{4\iota\pi r/V_1\tau} - c'e'f'\epsilon^{4\iota\pi r/V_2\tau}\right)\epsilon^{-2\iota\pi t/\tau},$$

whence, if $\lambda = \alpha(1-\alpha^2)$, $\mu = \beta(3\alpha^2 - 1)$,

the real parts become

$$-h = A(\lambda + q\mu)\cos\frac{2\pi}{\tau}\left(\frac{2r}{V_1} - t\right) + A(\lambda - q\mu)\cos\frac{2\pi}{\tau}\left(\frac{2r}{V_2} - t\right)$$

$$-g = -A(\lambda + q\mu)\sin\frac{2\pi}{\tau}\left(\frac{2r}{V_1} - t\right) + A(\lambda - q\mu)\sin\frac{2\pi}{\tau}\left(\frac{2r}{V_2} - t\right).$$

Let $$\phi = \frac{2\pi}{\tau}\left(\frac{2r}{V_2} - t\right), \quad \eta = \frac{2\pi r}{\tau}\left(\frac{1}{V_1} - \frac{1}{V_2}\right) \quad\text{.........(42),}$$

then $$-h = 2A\lambda\cos(\phi + \eta)\cos\eta - 2Aq\mu\sin(\phi + \eta)\sin\eta$$

$$-g = -2A\lambda\cos(\phi + \eta)\sin\eta - 2Aq\mu\sin(\phi + \eta)\cos\eta.$$

Let the analyser be placed in the position of extinction, and be then turned through a small angle ϵ, which will be considered positive when the analyser is turned towards the *right hand* of an observer who is looking through it; then the vibration on emerging from the analyser is

$$2A\lambda\cos(\phi + \eta)\sin(\eta - \epsilon) + 2Aq\mu\sin(\phi + \eta)\cos(\eta - \epsilon).$$

Since q^2 is to be neglected, the intensity of the emergent light is

$$4A^2\lambda^2(\sin\eta - \epsilon\cos\eta)^2 \quad\text{.....................(43),}$$

and will therefore be zero when

$$\tan\eta = \epsilon.$$

Now from (11), $$V_1^2 = U^2 - 2q$$

$$V_2^2 = U^2 + 2q$$

$$q = C\alpha/4\tau,$$

where C is Hall's constant, and α is the external magnetic force; whence the last of (42) becomes

$$\eta = \frac{\pi r C\alpha}{U^3\tau^2}.$$

Since C is negative for glass, it follows that η and therefore ϵ is negative; whence the plane of polarization is rotated in the *same* direction as the amperean currents, which agrees with Kundt's experimental result.

It also follows from (43) that the effect depends upon the direction of the current; for if the current is flowing in the same direction as that in which the analyser is rotated, its effect is to weaken the light, but if it flows in the opposite direction its effect is to strengthen the light.

Hence the effect produced by the glass plate is the *opposite* of that which is produced by a metallic magnetic pole.

486. The results obtained in the last two chapters, abundantly show the superiority of the electromagnetic theory over all other theories. This theory furnishes a satisfactory explanation of double refraction; and also of the photogyric properties of transparent media, when under the influence of magnetic force. The theory as far as it has hitherto been developed, does not account for ordinary and anomalous dispersion; and it fails to furnish a satisfactory explanation of metallic reflection, which is a phenomenon in all probability closely allied to the selective absorption, produced by substances which exhibit anomalous dispersion. We are therefore unable as yet, to give a complete theoretical explanation of the experiments of Kerr, on reflection from magnets; or of Kundt, on the transmission of light through thin magnetized metallic films. When electromagnetic waves travel through a medium which is susceptible to magnetic influence, the molecules of matter will be thrown into vibration; and the direction in which we ought to look for a theory, which will take cognizance of these hitherto unexplained phenomena, is one in which account is taken of the mutual reaction between ether and matter, and which will enable us to introduce the free periods of the matter vibrations into our equations.

THE END.

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